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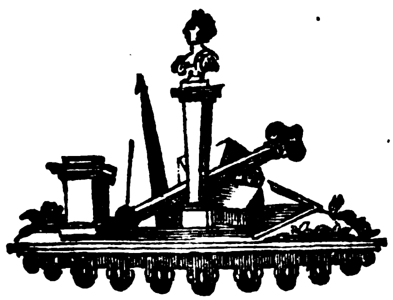
Opuscula analytica, tomus secundus

Leonhard Euler

LEONHARDI EVLERI

OPVSCVLA ANALYTICA.

Tomus secundus.



PETROPOLI

Typis Academiae Imperialis Scientiarum

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INDEX DISSERTATIONVM.

OPVS-

OPVSCVLA ANALYTICA.

CONSIDERATIONES
SVPER
THEOREMATE FERMATIANO
DE RESOLVTIONE NVMERORVM IN NVMEROS POLYGONALES.

§. 1.

Hoc theorema *Fermatianum* in se complectitur sequentes assertiones numero infinitas:

I. Omnem numerum esse summam trium trigonalium vel pauciorum.

II. Omnem numerum esse summam quatuor tetragonalium seu quadratorum, vel pauciorum.

III. Omnem numerum esse summam quinque pentagonalium, vel pauciorum.

IV. Omnem numerum esse summam sex hexagonalium, vel pauciorum.

V. Omnem numerum esse summam septem heptagonalium, vel pauciorum.

etc. etc.

 Quorum

Quorum theorematum cum *Fermatius* asseuerasset; demonstrationem a se esse inuentam, dubitari certe nequit, eius demonstrationem certissimis principiis fuisse innixam; ex quo eo magis dolendum est, eam post eius obitum prorsus periisse, vt nullum plane vestigium reperiri potuerit, cum sine dubio plerique Geometrae in his demonstrationibus inuestigandis frustra desudauerint. Hinc quidem excipienda est secunda assertio de resolutione numerorum in quatuor quadrata, cuius perfecta demonstratio ab Ingeniosissimo *La Grange* in lucem est protracta, quae autem ex eiusmodi principiis est deducta, vt inde nullum plane subsidium ad reliqua demonstranda expectari possit.

§. 2. Ingens igitur discrimen inter resolutionem in quadrata et reliquos numeros polygonales intercedere est censendum, quod potissimum in hoc consistit, quod resolutio in quaterna quadrata ad omnes plane numeros tam fractos quam integros se extendat, cum resolutio in alios polygonales tantum ad numeros integros restringatur, atque adeo nonnisi sub certa limitatione veritati sit consentanea. Resolutio enim in ternos trigonales manifesto tantum ad numeros integros adstringitur, cum infinitae dentur fractiones, quas nullo modo in ternas partes, in formula $\frac{xx+x}{2}$ contentas, resoluere licet; veluti si pro fractione $\frac{1}{2}$ statuere vellemus

$$\tfrac{1}{2} = \frac{xx+x}{2} + \frac{yy+y}{2} + \frac{zz+z}{2},$$

multiplicando per 8 deberet esse

$$4 = 4xx + 4x + 4yy + 4y + 4zz + 4z,$$

hincque tribus vnitatibus additis fieret

$$7 = (2x+1)^2 + (2y+1)^2 + (2z+1)^2.$$

Demon-

Demonſtratum autem eſt, numerum 7 nullo modo eſſe poſſe ſummam trium quadratorum. Quocirca ſi quis demonſtrationem huius partis ita inſtituere voluerit, vt, propoſito numero quocunque N, hanc aequationem:

$$N = \frac{xx+x}{2} + \frac{yy+y}{2} + \frac{zz+z}{2}$$

ſibi reſoluendam proponeret, oleum atque operam perdiderit.

§. 3. Porro vero etiam reſolutio in quinque pentagonales ad numeros integros adſtringitur, ſed longe alio modo atque in trigonalibus vſu venit. Nam ſi inter numeros pentagonales etiam fractiones in formula $\frac{3xx-x}{2}$ contentas admittere velimus, tum omnes plane numeros adeo in quatuor pentagonales diſcerpere liceret. Propoſito enim numero quocunque N ſi ſtatuamus

$$N = \frac{3xx-x}{2} + \frac{3yy-y}{2} + \frac{3zz-z}{2} + \frac{3vv-v}{2},$$

per 24 multiplicando fiet

$$24N = 36xx - 12x + 36yy - 12y + 36zz - 12z + 36vv - 12v$$

vnde quatuor vnitatibus additis fiet

$$24N + 4 = (6x-1)^2 + (6y-1)^2 + (6z-1)^2 + (6v-1)^2$$

Cum igitur numerus $24N+4$ certo ſit ſumma quatuor quadratorum, quae ſit $aa+bb+cc+dd$, hinc reperiemus

$$x = \frac{a+1}{6}, \quad y = \frac{b+1}{6}, \quad z = \frac{c+1}{6}, \quad v = \frac{d+1}{6}$$

atque horum radicum numeri pentagonales iunctim ſumti numero propoſito N aequabuntur. Nihilo vero minus ſi tantum numeros integros admittamus, vti *Fermatius* manifeſto poſtulat, vtique dantur eiusmodi numeri, quos in pauciores quam quinque pentagonales neutiquam reſoluere licet.

§. 4. Praeterea vero, etiamsi hinc fractiones excludamus et tantum numeros integros admittere velimus, tamen nouam limitationem adiicere, atque ex ordine pentagonalium omnes eos, quorum radices sunt numeri negatiui, excludere debemus. Cum enim formula generalis numerorum pentagonalium $\frac{3xx-x}{2}$ pro radicibus x negatiue sumptis praebeat hos numeros: 2, 7, 15, 26, 40, etc. si etiam hos admittere vellemus, non amplius quinque, sed tantum tres numeri pentagonales sufficerent omnibus plane numeris producendis, atque talis gemina limitatio multo magis pro sequentibus numeris polygonalibus est necessaria, vt theoremata *Fermatiana* veritati sint consentanea, quae limitatio sine dubio in caussa est, quod nulli adhuc Geometrae post *Fermatium* ad demonstrationem horum casuum penetrare licuerit.

§. 5. Cum igitur in demonstrationibus, quae desiderantur, harum restrictionum ratio necessario sit habenda, ipsa Theoremata *Fermatiana* sub alia forma repraesentemus, quae istas limitationes iam in se contineat, quod commodissime sequenti modo fieri posse videtur.

Pro resolutione in numeros trigonales.

§. 6. Consideretur series potestatum

$$1 + x + x^3 + x^6 + x^{10} + x^{15} + x^{21} + x^{28} + x^{36} + x^{45}$$

in qua potestates ipsius x progrediuntur secundum ipsos numeros trigonales

0, 1, 3, 6, 10, 15, 21, 28, etc.

ac primo manifestum est, si huius seriei quadratum capiatur, alias potestates ipsius x occurrere non posse, nisi quarum exponentes sint summae duorum numerorum trigonalium. Eodem

Eodem modo intelligitur, si eiusdem seriei capiatur cubus, in eo alias potestates non occurrere, nisi quarum exponentes sint summae trium numerorum trigonalium. Quocirca primum theorema *Fermatii* huc reducitur, vt si pro cubo assumtae seriei statuatur haec series per omnes potestates ipsuis x ascendens:

$$1 + Ax + Bxx + Cx^3 + Dx^4 + Ex^5 + Fx^6 + Gx^7 + Hx^8 + \text{etc.}$$

demonstretur in hac serie nullum plane coefficientem nihilo fore aequalem. Ex ipsa autem formatione manifestum est, nullum horum coefficientium fieri posse negatiuum.

Pro resolutione in numeros tetragonales, seu quadratos.

§. 7. Hic consideretur ista series potestatum:

$$P = 1 + x + x^4 + x^9 + x^{16} + x^{25} + x^{36} + \text{etc.}$$

cuius biquadratum P^4, si euoluatur, omnes complectetur potestates ipsius x, quarum exponentes sint summae quatuor quadratorum; atque adeo cuiusque coefficiens ostendet, quot variis modis exponens ipsius x in quatuor quadrata distribui queat; quare pro hoc casu demonstrari oportet, si ponatur

$$P^4 = 1 + Ax + Bxx + Cx^3 + Dx^4 + Ex^5 + Fx^6 + \text{etc.}$$

quae scilicet series per omnes potestates ipsuis x ascendat, nullum coefficientium A, B, C, D, E esse euaniturum. Hoc enim, si fuerit demonstratum, simul erit euictum, omnes plane numeros in quatuor quadrata resolui posse.

Pro

Pro reſolutione in numeros pentagonales.

§. 8. Conſideretur ſeries poteſtatum ipſius x, quarum exponentes ſecundum numeros pentagonales aſcendant, quae ſit

$$P = 1 + x + x^5 + x^{12} + x^{22} + x^{35} + x^{51} + \text{etc.}$$

eiusque euoluatur poteſtas quinta, quae ſit

$$P^5 = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + \text{etc.}$$

per omnes plane poteſtates ipſius x aſcendens, ac demonſtrandum eſt, in hac ſerie nullum prorſus coefficientem reperiri, qui ſit nihilo aequalis.

Pro reſolutione in numeros polygonales quoscunque.

§. 9. Sit π numerus laterum polygonorum, et cum formula generalis omnes iſtos numeros polygonales complectens ſit $= \frac{1}{2}(\pi - 2)zz - \frac{1}{2}(\pi - 4)z$, poſita ſcilicet radice $= z$, ſint omnes numeri polygonales hinc ordine reſultantes $0, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$, etc. vbi quidem conſtat eſſe

$$\alpha = 1,\ \beta = \pi,\ \gamma = 3\pi - 3,$$
$$\delta = 6\pi - 8,\ \varepsilon = 10\pi - 15,\ \zeta = 15\pi - 24;$$

tum vero conſideretur ſeries infinita

$$P = 1 + x^\alpha + x^\beta + x^\gamma + x^\delta + x^\varepsilon + x^\zeta + \text{etc.}$$

huiusque ſeriei ſumatur poteſtas exponentis $= \pi$, quae ſit

$$P^\pi = 1 + Ax + Bxx + Cx^3 + Dx^4 + Ex^5 + \text{etc.}$$

per omnes plane poteſtates ipſius x aſcendens, ac demonſtrandum erit, in hac ſerie nullum occurrere coefficientem nihilo aequalem; vnde patet, ſi modo hoc demonſtrari in genere poſſet, ſimul omnia theoremata *Fermatiana* fore demonſtrata.

§. 10.

§. 10. In hoc igitur negotio non parum fortaſſe proderit, ſi euolutionem iſtius poteſtatis P^{π} in genere docuero, ſimulque oſtendero, quomodo omnes coefficientes ſeriei euolutae a praecedentibus pendeant, ex iisque determinentur. Hunc in finem coefficientes ſingularum poteſtatum ipſius x idoneis characteribus deſignemus, ſitque poteſtatis x^{n} coefficiens $[n]$; quandoquidem hoc modo ſtatim perſpicitur, ad quamnam poteſtatem quisque coefficiens referatur. Nunc igitur inueſtigemus, quomodo quilibet coefficiens $[n]$ ex praecedentibus, qui ſunt $[n-1]$, $[n-2]$, $[n-3]$, $[n-4]$, etc. determinetur. Hoc enim modo iudicium facillime inſtituetur, num quis coefficiens nihilo aequalis fieri poſſit, id quod forſan eo facilius oſtendi poterit, cum certum ſit, nullum coefficientem fieri poſſe negatiuum.

§. 11. Cum igitur poſito

$$P = 1 + x^{\alpha} + x^{\beta} + x^{\gamma} + x^{\delta} + x^{\epsilon} + \text{etc.}$$

quaeri debeat iſtius ſeriei poteſtas exponentis π, ſtatuamus $S = P^{\pi}$, eritque ſumtis logarithmis $l\,S = \pi\, l\,P$, hincque differentiando $\frac{dS}{S} = \frac{\pi\, dP}{P}$, vnde formetur iſta aequatio:

$$P.\ \frac{x\,dS}{dx} = \frac{\pi S\,x\,dP}{dx},$$

pro qua ergo erit

$$\frac{x\,dP}{dx} = \alpha x^{\alpha} + \beta x^{\beta} + \gamma x^{\gamma} + \delta x^{\delta} + \varepsilon x^{\varepsilon} + \zeta x^{\zeta} + \text{etc.}$$

quae ergo ſeries per λS multiplicata idem productum generare debet, quod oritur, ſi ipſa ſeries

$$P = 1 + x^{\alpha} + x^{\beta} + x^{\gamma} + x^{\delta} + \text{etc.}$$

per formulam $\frac{x\,dS}{dx}$ multiplicetur.

§. 12. Ponatur igitur secundum characteres ante descriptos:

$$R = 1 + [1]x + [2]x^2 + [3]x^3 [4]x^4 + \text{etc.}$$

vbi notetur, primum terminum 1 aequiualere termino [0], vnde fiet

$$\frac{x\,dS}{dx} = 1[1]x^1 + 2[2]x^2 + 3[3]x^3 + 4[4]x^4 + \text{etc.}$$

hisque seriebus constitutis perpendamus, quot modis proposita potestas x^n in vtroque producto $P\frac{x\,dS}{dx}$ et $\pi\frac{S\,x\,dP}{dx}$ occurrat.

§. 13. Cum igitur ambo multiplicatores P et $\frac{x\,dP}{dx}$ alias potestates ipsius x non contineant, nisi quarum exponentes sunt α, β, γ, δ, etc. manifestum est, seriei S terminum $[n]x^n$ ex aliis terminis praecedentibus resultare non posse, praeter hos:

$$[n-\alpha]x^\alpha,\ [n-\beta]x^\beta,\ [n-\gamma]x^\gamma;$$

quamobrem omissis reliquis terminis consideremus tantum istos, ita vt habeamus:

$$S = [n]x^n \ldots + [n-\alpha]x^{n-\alpha} \ldots$$
$$+ [n-\beta]x^{n-\beta} \ldots + [n-\gamma]x^{n-\gamma} \ldots \text{etc.}$$

vbi per se manifestum est, hos terminos tantum eo vsque continuari debere, quoad exponentes $[n-\alpha]$, $[n-\beta]$, $[n-\gamma]$, etc. non fiunt negatiui. Multiplicetur igitur ista forma per

$$\frac{\pi\,x\,dP}{dx} = \pi\alpha x^\alpha + \pi\beta x^\beta + \pi\gamma x^\gamma + \pi\delta x^\delta + \text{etc.}$$

et productum sequentes terminos potestatem x^n continentes suppeditabit:

$$\pi\alpha[n-\alpha]x^n + \pi\beta[n-\beta]x^n + \pi\gamma[n-\gamma]x^n + \text{etc.}$$

§. 14.

§. 14. Deinde iisdem terminis retentis, erit

$$\frac{x\,dS}{dx} = n[n]x^n \ldots + (n-\alpha)[n-\alpha]x^{n-\alpha} \ldots + (n-\beta)[n-\beta]x^{n-\beta} \ldots$$
$$+ (n-\gamma)[n-\gamma]x^{n-\gamma} \ldots + (n-\delta)[n-\delta]x^{n-\delta} \ldots + \text{etc.}$$

quae forma ducta in seriem

$$P = 1 + x^\alpha + x^\beta + x^\gamma + x^\delta + \text{etc.}$$

sequentes praebebit terminos potestatem x^n continentes:

$$n[n]x^n + (n-\alpha)[n-\alpha]x^n + (n-\beta)[n-\beta]x^n$$
$$+ (n-\gamma)[n-\gamma]x^n + \text{etc.}$$

Quamobrem cum hoc productum priori debeat esse aequale, obtinebimus sequentem aequationem:

$$n[n] + (n-\alpha)[n-\alpha] + (n-\beta)[n-\beta] + (n-\gamma)[n-\gamma] + \text{etc.}$$
$$= \pi\alpha[n-\alpha] + \pi\beta[n-\beta] + \pi\gamma[n-\gamma] + \pi\delta[n-\delta] + \text{etc.}$$

§. 15. Hinc igitur patet, coefficientem $[n]$ ab iis tantum praecedentium, quorum characteres sunt

$$[n-\alpha],\ [n-\beta],\ [n-\gamma],\ \text{etc.}$$

pendere, ita vt sit

$$n[n] = \frac{\pi\alpha}{(n-\alpha)}[n-\alpha] \pm \frac{\pi\beta}{(n-\beta)}[n-\beta] \pm \frac{\pi\gamma}{(n-\gamma)}[n-\gamma] \pm \text{etc.}$$

Vbi quidem primo non est metuendum, ne ob terminos negatiuos vnquam valor ipsius $[n]$ proditurus sit negatiuus, quandoquidem hoc naturae rei repugnaret; et quia valor ipsius $[n]$ certe est numerus integer, euidens est omnes terminos in dextro membro iunctim sumtos semper valorem praebere debere vel n, vel $2n$, vel $3n$, vel $4n$, vel etc. nisi forte inde prodeat 0, quod igitur demonstrandum est nunquam eueniri posse, si quidem theorema *Fermatianum* fuerit veritati consentaneum, atque literae α, β, γ, δ, etc.

 deno-

denotent ordine omnes numeros polygonales pro numero laterum $= \pi$.

§. 16. Manifestum autem est, hanc determinationem coefficientis $[n]$ maxime esse generalem, neque tantum ad numeros polygonales extendi. Quaecunque enim series numerorum pro literis α, β, γ, δ, etc. accipiatur, aequatio inuenta semper habebit locum, et coefficiens $[n]$ indicabit quot variis modis numerus n possit esse summa π terminorum istius seriei:

$$0,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \eta,\ \text{etc.},$$

hincque numerus π ita definiri poterit, vt nullus huiusmodi coefficiens $[n]$ prodeat nihilo aequalis. Euidens enim est numerum π semper tantum accipi posse, vt dextrum membrum nostrae aequationis nunquam prodeat negatiuum. Quaeri autem semper solet minimus valor, qui pro π assumtus hoc sit praestaturus; vnde patet hanc methodum ad infinitas alias quaestiones huius generis pari fortasse successu applicari posse.

§. 17. Caeterum in genere notasse iuuabit, nisi fuerit $\alpha = 1$, semper fore $[n] = 0$, quamdiu fuerit $n < \alpha$; vnde in omnibus huiusmodi quaestionibus necesse est vt sit $\alpha = 1$, quem quidem terminum semper praecedere solet terminus $= 0$, siquidem series pro P assumta ab vnitate incipiat.

Applicatio ad numeros trigonales.

§. 18. Quo natura aequationis inuentae clarius percipiatur, eam ad numeros trigonales applicemus, pro quibus erit $\alpha = 1$, $\beta = 3$, $\gamma = 6$, $\delta = 10$, $\varepsilon = 15$, etc. tum vero, sumto

sumto exponente $\pi=3$, nostra formula inuenta hanc induet formam:

$$n[n]=(4-n)[n-1]+(12-n)[n-3]$$
$$+(24-n)[n-6]+(40-n)[n-10]+\text{etc.}$$

Hinc ergo, a numeris minimis incipiendo, sequentes nanciscemur reductiones:

$1\,[1]=3[0]$, ergo $[1]=3$

$2\,[2]=2[1]=6$, ergo $[2]=3$

$3\,[3]=1[2]+9[0]=12$, ergo $[3]=4$

$4\,[4]=0[3]+8[1]=24$, ergo $[4]=6$

$5\,[5]=-1[4]+7[2]=15$, ergo $[5]=3$

$6\,[6]=-2[5]+6[3]+18[0]=36$; ergo $[6]=6$

$7\,[7]=-3[6]+5[4]+17[1]=63$, ergo $[7]=9$

$8\,[8]=-4[7]+4[5]+16[2]=24$, ergo $[8]=3$

$9\,[9]=-5[8]+3[6]+15[3]=63$, ergo $[9]=7$

$10[10]=-6[9]+2[7]+14[4]+30[0]=90$, ergo $[10]=9$

$11[11]=-7[10]+1[8]+13[5]+29[1]=66$, ergo $[11]=6$

$12[12]=-8[11]+0[9]+12[6]+28[2]=108$, ergo $[12]=9$

$13[13]=-9[12]-1[10]+11[7]+27[3]=117$, ergo $[13]=9$

$14[14]=-10[13]-2[11]+10[8]+26[4]=84$, ergo $[14]=6$

$15[15]=-11[14]-3[12]+9[9]+25[5]+45[0]=90$, erg. $[15]=6$

$16[16]=-12[15]-4[13]+8[10]+24[6]+44[1]=240$, erg. $[16]=15$

$17[17]=-13[16]-5[14]+7[11]+23[7]+43[2]=153$, erg. $[17]=9$

$18[18]=-14[17]-6[15]+6[12]+22[8]+42[3]=126$, erg. $[18]=7$

$19[19]=-15[18]-7[16]+5[13]+21[9]+41[4]=228$, erg. $[19]=12$

$20[20]=-16[19]\;8[17]+4[14]+20[10]+40[5]=60$, erg. $[20]=3$

 Hinc

Hinc igitur patet, feriei $1 + x + x^3 + x^6 + x^{10} + x^{15} + \text{etc.}$ cubum euolui in hanc feriem:

$$1 + 3x^1 + 3x^2 + 4x^3 + 6x^4 + 3x^5 + 6x^6 + 9x^7$$
$$+ 3x^8 + 7x^9 + 9x^{10} + 6x^{11} + 9x^{12} + 9x^{13} + 6x^{14}$$
$$+ 6x^{15} + 15x^{16} + 9x^{17} + 7x^{18} + \text{etc.}$$

§. 19. In hac euolutione non iniucundum erat videre, quomodo priora membra negatiua a fequentibus pofitiuis femper fuperentur, atque exceffus femper per numerum n diuifibilis prodierit, id quod etiam perinde in forma generali neceffario vfu venire debet. Quod quo clarius in oculos incurrat, aequationem generalem fub hac forma exhibeamus:

$$n[n] = ((\pi+1)\alpha - n)[n-\alpha] + ((\pi+1)\beta - n)[n-\beta]$$
$$+ ((\pi+1)\gamma - n)[n-\gamma] \text{ etc.}$$

vnde partibus negatiuis ad finiftram translatis erit

$$n([n] + [n-\alpha] + [n-\beta] + [n-\gamma] + [n-\delta] + \text{etc.}) =$$
$$(\pi+1)(\alpha[n-\alpha] + \beta[n-\beta] + \gamma[n-\gamma] + \delta[n-\delta] + \text{etc.})$$

Hinc igitur intelligimus: primo fummam omnium horum valorum $\alpha[n-\alpha] + \beta[n-\beta] + \gamma[n-\gamma]$ femper fore diuifibilem per numerum n, nifi forte $\pi+1$ fuerit per n diuifibile; deinde fummam horum valorum

$$[n] + [n-\alpha] + [n-\beta] + [n-\gamma] + \text{etc.}$$

femper effe diuifibilem per numerum $\pi+1$, nifi forte ipfe numerus n per eum diuifionem admittat.

§. 20. Neque vero hae eximiae proprietates tantum in numeris polygonalibus, quos hic potiffimum contemplamur

mur, locum habent, ſed etiam generaliſſime obſeruari debent, quaecunque numerorum ſeries pro literis α, β, γ, etc. asſumatur, etiamſi ea nulli certae legi fuerit adſtricta. At vero haec ipſa circumſtantia non multum pro ſcopo noſtro polliceri videtur, quandoquidem theoremata *Fermatiana* tum demum veritati conſentanea cenſeri debent, quando pro litteris α, β, γ, δ, etc. ipſi numeri polygonales ordine ſcribuntur; ita vt, ſi earum ordo tantillum perturbaretur, demonſtratio etiam ipſa hinc petenda claudicare deberet; quocirca ad legitimam demonſtrationem horum theorematum inueniendam neceſſario opus erit, vt ſimul etiam ipſa lex progreſſionis literarum α, β, γ, δ, etc. in computum ducatur, id quod vtrum cum iſta methodo commode coniungi queat, nec ne, non tam facile perſpicitur. Interim tamen iſtae conſiderationes fortaſſe aliis aliquam lucem accendere poterunt, quo felicius ad has veritates penetrare valeant.

OBSER-

OBSERVATIONES
IN ALIQVOT
THEOREMATA
ILLVSTR. DE LA GRANGE.

Postquam aliquod Theorema, ex iis quae non ita pridem demonstraui, quo ostendi, formulae integralis $\int \frac{(x-1)dx}{lx}$, si post integrationem ponatur $x = 1$, valorem esse $= l2$, cum illustri Domino *de la Grange* communicassem, is novitate huius argumenti permotus, non solum felicissimo successu eius demonstrationem penetrauit, sed etiam plurima alia praeclara inuenta inde deduxit, quorum vberior enucleatio scientiae analyticae maxima incrementa polliceri videtur, ex quo genere aliquot praeclarissima specimina mecum beneuole communicauit, quae statim summo studio sum perscrutatus; et quoniam haec materia attentionem mereri videtur, meas meditationes, quae se mihi hac occasione obtulerunt, fusius sum expositurus. Cum autem hoc quasi nouum Analyseos genus potissimum in eiusmodi formulis integralibus versetur, in quibus variabili post integrationem certus valor determinatus tribuitur, ad taediosas verborum ambages euitandas, quas perpetua talium conditionum commemoratio postularet, peculiarem signandi modum adhibebo, quem ante omnia accuratius explicare necesse erit.

Hypo-

Hypothesis.

§. 1. Hac signandi ratione:

$$\int P\,dx \left[\begin{array}{l} \text{ab } x = a \\ \text{ad } x = b \end{array} \right]$$

declaratur, integrale $\int P\,dx$ ita esse assumtum, vt euanescat posito $x = a$, tum vero statui $x = b$; quo pacto manifestum est eius valorem penitus fore determinatum.

Scholion.

§. 2. Quo indoles huius determinationis clarius perspiciatur, quoniam P denotat functionem aliquam ipsius x, eius naturam repraesentemus linea quadam curua $i\,x\,a\,b\,c\,o$, super axe I O exstructa, cuius quaecunque applicata X x, abscissae I X $= x$ respondens, exhibeat ipsam functionem P, ita vt formula integralis $\int P\,dx$ indefinite exprimat aream huius curuae. Quod si iam capiantur abscissae I A $= a$, I B $= b$, quibus respondeant applicatae A a et B b, formula proposita exprimet aream A a B b, inter applicatas A a et B b interceptam. Eodem modo, si alia quaepiam abscissa statuatur I C $= c$ area A a C c exprimetur hac formula: Tab. I. Fig. 1.

$$\int P\,dx \left[\begin{array}{l} \text{ab } x = a \\ \text{ad } x = c \end{array} \right];$$

area autem B b C c ista formula:

$$\int P\,dx \left[\begin{array}{l} \text{ab } x = b \\ \text{ad } x = c \end{array} \right];$$

tum vero, ab initio I incipiendo, area I i A a indicabitur per hanc formulam:

$$\int P\,dx \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = a \end{array} \right],$$

vnde ſponte fluunt ſequentia lemmata ita ſuccincte expreſſa:

§. 3. *Lemma I.*

$$\int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix} = -\int P\,dx \begin{bmatrix} \text{ab } x = b \\ \text{ad } x = a \end{bmatrix}.$$

Quoniam enim, ſi b vt maius ſpectetur quam a, formula poſterior

$$\int P\,dx \begin{bmatrix} \text{ab } x = b \\ \text{ad } x = a \end{bmatrix}$$

eandem aream A a B b refert quam prior, ſed ordine retrogrado, iſta expreſſio pro negatiua erit habenda, ſicque erit quoque

$$\int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix} + \int P\,dx \begin{bmatrix} \text{ab } x = b \\ \text{ad } x = a \end{bmatrix} = 0.$$

§. 4. *Lemma II.*

$$\int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix} + \int P\,dx \begin{bmatrix} \text{ab } x = b \\ \text{ad } x = c \end{bmatrix} = \int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = c \end{bmatrix}$$

quemadmodum inſpectio figurae manifeſto declarat.

§. 5. *Lemma III.*

$$\int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = c \end{bmatrix} - \int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix} = \int P\,dx \begin{bmatrix} \text{ab } x = b \\ \text{ad } x = c \end{bmatrix};$$

vbi in binis prioribus formulis idem occurrit terminus *a quo*, ſcilicet $x = a$, terminorum vero *ad quem*, ſcilicet $x = c$ et $x = b$, poſterior $x = b$ dat pro tertia formula terminum *a quo*, prior vero terminum *ad quem*.

§. 6.

§. 6. *Lemma IV.*

$$\int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = c \end{bmatrix} - \int P\,dx \begin{bmatrix} \text{ab } x = b \\ \text{ad } x = c \end{bmatrix} = \int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix};$$

vbi notetur, binas formulas priores eundem habere terminum *ad quem*, scilicet $x = c$, terminorum autem *a quo* priorem $x = a$ dare in tertia terminum *a quo*, posteriorem vero terminum *ad quem*.

§. 7. *Lemma V.*

$$\int P\,dx \begin{bmatrix} \text{ab } x = a \\ \text{ad } x = b \end{bmatrix} + \int P\,dx \begin{bmatrix} \text{ab } x = b \\ \text{ad } x = c \end{bmatrix} + \int P\,dx \begin{bmatrix} \text{ab } x = c \\ \text{ad } x = a \end{bmatrix} = 0.$$

Scholion.

§. 8. His igitur, quae per se sunt maxime perspicua, praemissis, argumenta praecipua, quae celeb. *de la Grange* mihi perscripsit ordine percurram. Primo autem mentionem insignis paradoxi facit, cuius indolem ipse non satis perspicere fatetur, a quo igitur meas meditationes inchoabo.

Resolutio insignis Paradoxi.

§. 9. Cum Vir celeb. etiam inuenisset hoc theorema generale

$$\int \frac{x^n - x^m}{l x} \frac{dx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \frac{n}{m}$$

cuius veritatem non ita pridem pluribus demonstrationibus adstruxi, posuit $x^n = z$ et $x^m = y$: quo facto pars prior $\int \frac{x^{n-1}\,dx}{l x}$ transformatur in hanc: $\int \frac{dz}{lz}$; simili vero modo altera $\int \frac{x^{m-1}\,dx}{lx}$ in hanc: $\int \frac{dy}{ly}$; vnde his partibus feorfim pofitis fequitur fore

$$\int \frac{dz}{lz} \begin{bmatrix} \text{a} & z=0 \\ \text{ad} & z=1 \end{bmatrix} - \int \frac{dy}{ly} \begin{bmatrix} \text{ab} & y=0 \\ \text{ad} & y=1 \end{bmatrix} = l\frac{n}{m}.$$

Quare cum hae duae formulae omnino fint fimiles, atque iisdem terminis integrationis contentae, quis non crederet eos etiam inter fe perfecte fore aequales, fiue effe

$$\int \frac{dz}{lz} \begin{bmatrix} \text{a} & z=0 \\ \text{ad} & z=1 \end{bmatrix} = \int \frac{dy}{ly} \begin{bmatrix} \text{ab} & y=0 \\ \text{ad} & y=1 \end{bmatrix}?$$

Interim tamen vidimus, differentiam inter has formulas effe $l\frac{n}{m}$. Hic igitur fe offert quaeftio maximi momenti: quemadmodum iftam manifeftam contradictionem dirimere oporteat?

§. 10. Primo autem hic obferuari conuenit, ambas quantitates y et z certo quodam modo a fe inuicem pendere. Cum enim fit $y = x^m$ et $z = x^n$, erit $y^n = z^m$, quo tamen nexu non impeditur, quo minus, pofito fiue $y = 0$, fiue $y = 1$, etiam fiat $z = 0$, fiue $z = 1$. Interim tamen hinc neutiquam patet, cur ob hanc rationem iftae binae formulae:

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab} & y=0 \\ \text{ad} & y=1 \end{bmatrix} \text{ et } \int \frac{dz}{lz} \begin{bmatrix} \text{a} & z=0 \\ \text{ad} & z=1 \end{bmatrix},$$

difpares prodire queant; vnde haec obferuatio ad dubium foluendum nihil plane conferre videtur.

§. 11. Quin etiam nullo prorfus dubio obnoxia videtur haec aequatio multo generalior:

$$\int \frac{dy}{ly} \begin{bmatrix} \text{ab} & y=a \\ \text{ad} & y=b \end{bmatrix} = \int \frac{dz}{lz} \begin{bmatrix} \text{a} & z=a \\ \text{ad} & z=b \end{bmatrix};$$

quando-

quandoquidem nihil plane impedit, quo minus loco z scribamus y, vel viciſſim; verum plurima phaenomena in analyſi obſeruata ſatis luculenter docent, huiusmodi aequalitates interdum exceptionem pati, quando valores euadunt infiniti. Haec autem circumſtantia noſtro caſu vtique locum habet, cum formula integralis $\int\frac{dy}{ly}$, ſi ab $y=0$ ad $y=1$ extendatur, vtique in infinitum excreſcat, quod etiam de altera: $\int\frac{dz}{lz}$, eſt tenendum. Si enim fiat $=1$, applicata noſtrae curuae, quae eſt $\frac{1}{lz}$, manifeſto fit infinite magna, vnde ſuperior aequalitas generalis

$$\int\frac{dy}{ly}\left[\begin{array}{l}\text{ab } y=a\\ \text{ad } y=b\end{array}\right]-\int\frac{dz}{lz}\left[\begin{array}{l}\text{ab } z=a\\ \text{ad } z=b\end{array}\right]=0$$

hanc reſtrictionem poſtulat, niſi vel a ſit $=1$, vel $b=1$, quippe quibus caſibus vtraque formula fit infinita.

§. 12. His perpenſis nullum plane dubium mihi quidem ſupereſſe videtur, quin in hac circumſtantia vera ſolutio propoſiti paradoxi ſit quaerenda, quae ſcilicet in eo verſatur, quod ſit tam $\int\frac{dy}{ly}\left[\begin{array}{l}\text{ab } y=0\\ \text{ad } y=1\end{array}\right]=\infty$, quam $\int\frac{dz}{lz}\left[\begin{array}{l}\text{a } z=0\\ \text{ad } z=1\end{array}\right]=\infty$, ita vt horum infinitorum differentia poſſit aequari quantitati finitae cuicunque, ideoque in ſe ſpectata prorſus non determinetur; quod autem iſta differentia noſtro caſu ſit $l\frac{n}{m}$, ideoque determinata, inde venit quod ſit $y^n=z^m$.

§. 13. Simile aliquid euenire poteſt in formulis ſimplicioribus, quales ſunt $\int\frac{dy}{y}$ et $\int\frac{dz}{z}$, quippe quarum valores, a termino $y=0$ et $z=0$ ſumti, ſunt infiniti, vnde etiamſi

 poſt

post integrationem idem terminus ad quem statuatur, scilicet $y = 1$ et $z = 1$, tamen hinc nullo modo sequitur, differentiam absolute nihilo aequari, quin potius tanquam indeterminata spectari debebit, cum quidem pro aliis terminis integrationis certo sit

$$\int \frac{dy}{y} \left[\begin{array}{l} \text{ab } y = a \\ \text{ad } y = b \end{array} \right] = \int \frac{dz}{z} \left[\begin{array}{l} \text{a } \; z = a \\ \text{ad } z = b \end{array} \right],$$

dummodo neque a neque b fuerit $= 0$ vel $= \infty$.

§. 14. Atque hinc etiam paradoxon proposito penitus simile proferri potest, quod ita se habet:

$$\int \frac{dz}{z} \left[\begin{array}{l} \text{a } \; z = 0 \\ \text{ad } z = \infty \end{array} \right] - \int \frac{dy}{y} \left[\begin{array}{l} \text{ab } y = 0 \\ \text{ad } y = \infty \end{array} \right] = l\,a,$$

cuius veritas cum in aprico sit posita, si quidem accipiatur $z = a\,y$, etiam paradoxon propositum rite dilutum erit censendum.

Obseruationes in hoc Theorema D. de la Grange.

$$\int \frac{(x^n - x^m)}{l\,x} \cdot \frac{dx}{x} \left[\begin{array}{l} \text{ab } x = a \\ \text{ad } x = b \end{array} \right] = \int (b^y - a^y) \frac{dy}{y} \left[\begin{array}{l} \text{ab } y = m \\ \text{ad } y = n \end{array} \right].$$

§. 15. Cum equidem ante aliquod tempus reductiones huiusmodi formularum tractassem, alios terminos integrationis, praeterquam ab $x = 0$ ad $x = 1$, non sum contemplatus, vnde hoc Theorema mihi statim altioris indaginis est visum, atque omnino dignum quod summa cura expendatur. Primum igitur in eius veritatem per series inquirere constitui, quod negotium sequenti modo peregi.

§. 16.

§. 16. Cum fit

$$x^\alpha = e^{\alpha l x} = 1 + \alpha l x + \frac{(\alpha l x)^2}{1.2} + \frac{(\alpha l x)^3}{1.2.3} \text{ etc. erit}$$

$$x^n - x^m = (n-m)\frac{lx}{1} + (n^2 - m^2)\frac{(lx)^2}{1.2} + \frac{(n^3 - m^3)(lx)^3}{1.2.3} + \text{etc.}$$

Hanc ergo feriem ducamus in $\frac{dx}{xlx}$, et quia in genere

$$\int (lx)^\lambda \frac{dx}{xlx}\left[\begin{array}{l}\text{ab } x = a\\ \text{ad } x = b\end{array}\right] = \frac{(lb)^\lambda - (la)^\lambda}{\lambda}$$

formulae ad finiftram partem fcriptae valor per hanc feriem infinitam exprimetur:

$$\frac{(n-m)}{1}\frac{(lb-la)}{1} + \frac{(n^2-m^2)}{1.2}\frac{(lb)^2-(la)^2}{2} + \frac{(n^3-m^3)}{1.2.3}\frac{(lb)^3-(la)^3}{3} + \text{etc.}$$

§. 17. Simili modo pro formula ad dextram pofita per feriem infinitam erit

$$b^y - a^y = y\frac{(lb-la)}{1} + y^2\frac{(lb)^2-(la)^2}{1.2} + y^3\frac{(lb)^3-(la)^3}{1.2.3}$$

quae ergo ducatur in $\frac{dy}{y}$, et quia in genere eft

$$\int y^\lambda \frac{dy}{y}\left[\begin{array}{l}\text{ab } y = m\\ \text{ad } y = n\end{array}\right] = \frac{n^\lambda - m^\lambda}{\lambda},$$

valor iftius formulae per feriem hanc infinitam exprimetur:

$$\frac{(n-m)}{1}\frac{(lb-la)}{1} + \frac{(n^2-m^2)}{2}\frac{(lb^2-la^2)}{1.2} + \frac{(n^3-m^3)}{3}\frac{(lb^3-la^3)}{1.2.3} + \text{etc.}$$

Quia igitur haec feries cum praecedente perfecte congruit, veritas theorematis firmiter eft euicta.

§. 18. Verum hinc neutiquam perfpicitur, quomodo fagaciffimus auctor ad hoc Theorema fit perductus, quam obrem, rebus probe perpenfis, vifum inueni, ex iisdem principiis, quibus antehac fum vfus, ad easdem formulas perueniendi. Inchoandum autem eft ab hac forma fimpliciffima:

$\int x$

$$\int x^{\lambda}\frac{dx}{x}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix}=\frac{b^{\lambda}-a^{\lambda}}{\lambda},$$

vbi vtrinque per $d\lambda$ multiplicans denuo integrationem instituo, et cum, vti iam passim demonstratum reperitur, sit

$$\int d\lambda\int x^{\lambda}\tfrac{dx}{x}=\int\tfrac{dx}{x}\int x^{\lambda}\,d\lambda,$$

quaeri tantum debet hoc integrale: $\int x^{\lambda}\,d\lambda$, spectata quantitate x vt constante, ita vt sola λ sit variabilis. Est vero

$$\int x^{\lambda}\,d\lambda=\frac{x^{\lambda}}{lx}+\text{C},$$

quemadmodum ex elementis calculi exponentialis liquet. Hic vero cardo rei in hoc versatur, vt istud integrale certa lege definiatur, quam deinceps etiam in altera parte obseruari oportet. Statuamus ergo talia integralia ita capi, vt euanescant posito $\lambda=0$, eritque

$$\int x^{\lambda}\,d\lambda=\frac{x^{\lambda}-1}{lx},$$

quo pacto pro sinistra parte habebimus

$$\int d\lambda\int x^{\lambda}\frac{dx}{x}=\int\frac{dx}{x}\frac{(x^{\lambda}-1)}{lx}.$$

§. 19. Pro parte autem dextra habebimus

$$\int\tfrac{d\lambda}{\lambda}(b^{\lambda}-a^{\lambda}),$$

qua formula eadem lege integrata, vt facto $\lambda=0$ prodeat nihilum, hunc valorem more hic recepto repraesentare licebit:

$$\int\frac{dy}{y}(b^{y}-a^{y})\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=\lambda\end{bmatrix}.$$

Hic

Hic enim nil aliud fecimus, nisi quod pro λ scripsimus y, et facta integratione loco y eius valorem λ restitui assumsimus, sicque assecuti sumus sequentem formulam:

$$\int (x^\lambda - 1)\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix} = \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=\lambda\end{bmatrix},$$

quam tanquam Theorema vtilissimum spectare licet.

§. 20. Vi ergo huius Theorematis nanciscimur sequentes reductiones:

$$\int (x^n - 1)\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix} = \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=n\end{bmatrix} \text{ et}$$

$$\int (x^m - 1)\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix} = \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=m\end{bmatrix};$$

quare si formula posterior a priore subtrahatur, erit

$$\int (x^n - x^m)\frac{dx}{x\,lx}\begin{bmatrix}\text{ab } x=a\\ \text{ad } x=b\end{bmatrix} = \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=n\end{bmatrix}$$

$$- \int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=0\\ \text{ad } y=m\end{bmatrix};$$

verum ista formula ad dextram posita per reductionem in lemmate 3° ostensam reuocatur ad hanc formam simpliciorem:

$$\int \frac{dy}{y}(b^y - a^y)\begin{bmatrix}\text{ab } y=m\\ \text{ad } y=n\end{bmatrix};$$

vnde patet, hoc modo ipsum hoc insigne Theorema etiam ex nostris principiis inuestigari potuisse.

§. 21. Hoc autem Theoremate generalissimo vir ingeniosissimus est vsus ad Theorema meum demonstrandum, quo ostendi esse

$$\int (x^n - x^m) \frac{dx}{x\,lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array} \right] = l\frac{n}{m};$$

tantum enim opus erat, vt caperetur $a = 0$ et $b = 1$, quo pacto formula ad dextram posita integralis abit in

$$\int \frac{dy}{y} \left[\begin{array}{l} \text{ab } y = m \\ \text{ad } y = n \end{array} \right],$$

cuius valor manifesto fit $ln - lm = l\frac{n}{m}$, quae est noua demonstratio mei Theorematis, cuiusmodi quidem dudum plures alias dederam.

Obseruationes in Theorema D. de la Grange.

$$\int \frac{x^n - x^m\, dx}{(1 + x^r)\, lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array} \right] = l\left(\frac{\text{tag.}\, \frac{(n+1)\pi}{2r}}{\text{tag.}\, \frac{(m+1)\pi}{2r}} \right).$$

§. 22. Quia hic ambo exponentes m et n neque a se inuicem neque ab exponente r pendent, manifestum est, pro vtraque potestate x^m et x^n seorsim integrale talem formam habere debere:

$$\int \frac{x^n\, dx}{(1 + x^r)\, lx} = l\, \text{tag.}\, \frac{(m+1)\pi}{2r} + C \text{ et}$$

$$\int \frac{x^m\, dx}{(1 + x^r)\, lx} = l\, \text{tag.}\, \frac{(m+1)\pi}{2r} + C.$$

Si enim posterior forma a priore subtrahatur, constans C ex ex calculo egreditur, et ipsum integrale propositum resultat. Hic igitur plurimum intererit valorem istius constantis C determinasse.

§. 23. Inter formulas integrales, quarum valores pro casu, quo post integrationem variabilis infinita statuitur, ex primis principiis calculi integralis assignaui, reperitur ista:

$$\int \frac{x^{k+n}}{1+x^{2k}} \frac{dx}{x} \left[\begin{array}{l}\text{ab } x=0 \\ \text{ad } x=\infty\end{array}\right] = \frac{\pi}{2k \operatorname{cos}. \frac{n\pi}{2k}} = \frac{\pi}{2k \operatorname{fin}. \frac{(k+n)\pi}{2k}}$$

vbi autem assumitur, exponentem n non maiorem capi quam k. Quod si iam hic exponens n vt variabilis tractetur, spectata ipsa x vt constante, et vtrinque per dn multiplicetur denuoque integretur, formula sinistra erit

$$\int dn \int \frac{x^{k+n}}{1+x^{2k}} \frac{dx}{x} = \int \frac{dx}{x(1+x^{2k})} \int x^{k+n}\, dn,$$

vbi postremum integrale fit

$$\int x^{k+n}\, dn = \frac{x^{k+n}}{lx} + C.$$

Vt autem hoc integrale determinetur, constantem ita definiamus, vt id euanescat posito $n=0$, vnde obtinetur

$$\int x^{k+n}\, dn = \frac{x^{k+n} - x^{k}}{lx},$$

ita vt formula integralis ad sinistram posita futura sit

$$\int \frac{x^{k+n} - x^{k}}{1+x^{2k}} \cdot \frac{dx}{x\, lx} \left[\begin{array}{l}\text{ab } x=0 \\ \text{ad } x=\infty\end{array}\right].$$

§. 24. Pro parte dextra autem habebimus hoc integrale: $\int \frac{\pi\, dn}{2k \operatorname{fin}. \frac{(k+n)\pi}{2k}}$, etiam ita sumendum, vt euanescat

posito

posito $n = 0$. Hunc in finem statuamus augulum $\frac{(k+n)\pi}{2k} = \phi$, et quia hinc erit $d\phi = \frac{\pi dn}{2k}$, formula nostra integranda erit $\int \frac{d\Phi}{\sin.\Phi}$, cuius integrale per regulas notas in genere est:

$$l\,\text{tag.}\,\tfrac{1}{2}\phi + C = l\,\text{tag.}\,\frac{(k+n)\pi}{4k} + C,$$

quod, facto $n = 0$, abit in $l\,\text{tag}\,\frac{\pi}{4} + C$. Quare cum tag. $\frac{\pi}{4} = 1$ et $l\,1 = 0$, euidens est constantem C fore $= 0$, ita vt integrale hoc quaesitum sit $l\,\text{tag.}\,\frac{(k+n)\pi}{4k}$. Hinc ergo assecuti sumus istam reductionem generalem:

$$\int \frac{x^{k+n} - x^k}{1 + x^{2k}} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = l\,\text{tag.}\,\frac{(k+n)\pi}{4k},$$

vbi autem probe notari oportet, exponentes m et n maiores capi non licere quam k.

§. 25. Cum igitur, loco n alium numerum m sumendo, simili modo fit:

$$\int \frac{x^{k+m} - x^k}{1 + x^{2k}} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = l\,\text{tag.}\,\frac{(k+m)\pi}{4k},$$

subtrahatur ista formula a praecedente, et obtinebitur ista:

$$\int \frac{x^{k+n} - x^{k+m}}{1 + x^{k}} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = l\left(\frac{\text{tag.}\,\frac{(k+n)\pi}{4k}}{\text{tag.}\,\frac{(k+m)\pi}{4k}}\right),$$

quae manifesto cum forma proposita congruit, si modo loco $k + n - 1$ scribatur n et m loco $k + m - 1$, at loco exponentis $2k$ scribatur r, tum enim manifesto fiet:

$$\int \frac{x^n - x^m}{(1 + x^r)} \cdot \frac{dx}{lx} \left[\begin{array}{l}\text{ab } x = 0\\ \text{ad } x = \infty\end{array}\right] = l\left(\frac{\text{tag.}\,\frac{(n+1)\pi}{2r}}{\text{tag.}\,\frac{(m+1)\pi}{2r}}\right).$$

§. 26.

§. 26. Quoniam ista analysis nos perduxit ad hanc formam:

$$\int \frac{x^{k+n} - x^k}{1 + x^{2k}} \cdot \frac{dx}{x\, lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = l \text{ tang. } \frac{(k+n)\pi}{4k},$$

hic maximi momenti erit obseruasse, semper fore

$$\int \frac{x^k}{1 + x^{2k}} \cdot \frac{dx}{x\, lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = 0$$

id quod ita ostendere possum: Ponatur $x^k = z$, erit

$$x^{k-1}\, dx = \frac{dz}{k} \text{ et } lx = \frac{lz}{k},$$

sicque ista formula induet hanc formam: $\int \frac{dz}{(1+zz)\, lz}$, vbi termini integrationis etiamnunc sunt $z = 0$ et $z = \infty$. Fiat porro $z = \text{tang.}\, \phi$, vnde termini integrationis erunt $\phi = 0$ et $\phi = \frac{\pi}{2}$; hinc autem ob $d\phi = \frac{dz}{1+zz}$ nascetur ista formula:

$$\int \frac{d\phi}{l \text{ tang. } \phi} \left[\begin{array}{l} \text{a } \phi = 0 \\ \text{ad } \phi = \frac{\pi}{2} \end{array}\right]$$

cuius valorem in nihilum abire ostendi debet.

§. 27. Ad hoc demonstrandum statuatur axis $I\Pi = \frac{\pi}{2}$, super quo ab initio I sumta abscissa indefinita $Ip = \phi$, applicata sit $= \frac{1}{l \text{ tang. } \phi}$. Quod si ergo hic axis $I\Pi$ in O bisecetur, vt sit $IO = \frac{\pi}{4}$, in hoc puncto applicata erit Tab. I. Fig. 2.

$$= \frac{1}{l \text{ tang. } \frac{\pi}{4}} = \infty.$$

Iam ab hoc puncto O vtrinque capiantur interualla aequalia $Op = Oq = \omega$, et pro puncto p erit $\phi = \frac{\pi}{4} - \omega$, sicque in hoc puncto p applicata erit $\frac{1}{l \text{ tang. } (\frac{\pi}{4} - \omega)}$; est vero ro tang. $(\frac{\pi}{4} - \omega) = \cot. (\frac{\pi}{4} + \omega)$, quare cum sit $l\cot. = -l\text{tang.}$ applicata in hoc puncto p erit $\frac{-1}{l\,\text{tang.}\,(\frac{\pi}{4} + \omega)}$; at quia est

$Iq = (\frac{\pi}{4} + \omega)$, erit applicata in puncto $q = \frac{+1}{l\,\text{tang.}\,(\frac{\pi}{4} + \omega)}$, sicque aequalis est applicatae in p, sed in contrarium vergens. Ita si applicata sursum directa fuerit qQ, in puncto p eadem applicata deorsum erit directa $pP = qQ$.

§. 28. Quod si ergo talis curua super axe $I\Pi = \frac{\pi}{2}$ exstruatur, ita vt abscissae ϕ respondeat applicata $\frac{1}{l\,\text{tang.}\,\phi}$, haec curua ex duabus portionibus inter se perfecte aequalibus constabit, circa punctum medium O ita dispositis, vt curva sinistra sit IPM in infinitum descendens ad asymptotam Om, pars autem dextra similimodo a Π sinistrorsum sursum ascendet ad asymptotam On. Quare cum formula integralis $\int \frac{d\Phi}{l\,\text{tang.}\,\Phi}$, a $\phi = 0$ ad $\phi = \frac{\pi}{2}$ extensa, exprimat totius huius curuae ab I vsque ad Π protensae aream, euidens est, totam hanc aream ad nihilum redigi, quia portio eius negatiue sumenda perfecte similis est portioni positiue sumendae.

§. 29. Sic igitur per demonstrationem omnino singularem euictum est, semper esse:

$$\int \frac{x^k}{1 + x^{2k}} \frac{dx}{x\,lx} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = 0,$$

quod certe est Theorema in hoc genere maxime notatu dignum. Quod si ergo cum illustri D. *de la Grange* statuamus $2k = r$, erit $\int \frac{x^{\frac{1}{2}r - 1}\,dx}{(1 + x^r)\,lx} = 0$; praeterea vero pro

nostra

nostra formula §. 24. exhibita, ob

$$\int \frac{x^k\, dx\, l}{(1+x^{2k})\, x\, l x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=\infty\end{matrix}\right]=0,$$

deducitur istud Theorema omnino notabile:

$$\int \frac{x^{k+n}}{1+x^{2k}}\frac{dx}{x\, l x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=\infty\end{matrix}\right]=l\,\text{tang.}\,\frac{(k+n)\pi}{4k},$$

quod more D. *de la Grange* ita proponi potest:

$$\int \frac{x^n\, dx}{(1+x^r)\, l x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=\infty\end{matrix}\right]=l\,\text{tang.}\,\frac{(n+1)\pi}{2r}$$

sicque patet constantem illam supra §. 22. a nobis inductam reuera nihilo aequari.

§. 30. Quoniam Demonstratio huius Theorematis methodo satis insueta innititur, eius veritatem per series ostendisse iuuabit. Ad hoc autem valorem formulae

$$\int \frac{x^{\lambda-1}\, dx}{(1+x^r)\, l x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=\infty\end{matrix}\right]$$

in duas partes diuelli necesse est (scilicet loco n scribendo $\lambda-1$), quae sint

$$P=\int \frac{x^{\lambda-1}\, dx}{(1+x^r)\, l x}\left[\begin{matrix}\text{ab } x=0\\ \text{ad } x=1\end{matrix}\right] \text{ et}$$

$$Q=\int \frac{x^{\lambda-1}\, dx}{(1+x^r)\, l x}\left[\begin{matrix}\text{ab } x=1\\ \text{ad } x=\infty\end{matrix}\right]$$

ita vt $P+Q$ exprimat valorem quem quaerimus. Nunc in posteriore parte loco x scribamus $\frac{1}{z}$, fietque

$$Q=\int \frac{z^{-\lambda}}{(1+z^{-r})}\frac{dz}{z\, l z}\left[\begin{matrix}\text{a } z=1\\ \text{ad } z=0\end{matrix}\right]=\int \frac{z^{r-\lambda}}{(1+z^r)}\frac{dz}{z\, l z}\left[\begin{matrix}\text{a } z=1\\ \text{ad } z=0\end{matrix}\right]$$

et

et commutatis terminis integrationis

$$Q = -\int \frac{z^{r-\lambda}}{1+z^r} \cdot \frac{dz}{z\,lz} \left[\begin{array}{l} \text{a } z=0 \\ \text{ad } z=1 \end{array}\right].$$

Nunc autem loco z scribamus x, quia termini integrationis vtrinque sunt iidem, erit

$$P+Q = \int \frac{x^{\lambda} - x^{r-\lambda}}{1+x^r} \cdot \frac{dx}{x\,lx} \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array}\right]$$

cuius ergo valor formulae propositae est aequalis.

§. 31. Iam fractionem $\frac{1}{1+x^r}$ in seriem infinitam conuertamus

$$1 - x^r + x^{2r} - x^{3r} + x^{4r} - \text{etc.}$$

cuius singuli termini in $\frac{dx}{x\,lx}(x^{\lambda} - x^{r-\lambda})$ ducti producunt

$$\frac{dx}{x\,lx}(x^{\lambda} - x^{r-\lambda}) - \frac{dx}{x\,lx}(x^{r+\lambda} - x^{2r-\lambda}) + \frac{dx}{x\,lx}(x^{2r+\lambda} - x^{3r-\lambda})$$
$$- \frac{dx}{x\,lx}(x^{3r+\lambda} - x^{4r-\lambda}) + \text{etc.}$$

Cum autem per Theorema principale in hoc genere sit

$$\int \frac{dx}{x\,lx}(x^{\alpha} - x^{\gamma}) \left[\begin{array}{l} \text{ab } x=0 \\ \text{ad } x=1 \end{array}\right] = l\frac{\alpha}{\beta},$$

singulis membris hoc modo integratis prodibit

$$P+Q = l\frac{\lambda}{r-\lambda} - l\frac{r+\lambda}{2r-\lambda} + l\frac{2r+\lambda}{3r-\lambda} - l\frac{3r+\lambda}{4r-\lambda} + \text{etc.}$$

§. 32. Omnes hos logarithmos in vnicum compingere licebit, ratione habita signi cuiusque, hocque modo reperietur fore

$$P+Q = l\frac{\lambda}{r-\lambda} \cdot \frac{3r-\lambda}{r+\lambda} \cdot \frac{2r+\lambda}{3r-\lambda} \cdot \frac{4r-\lambda}{3r+\lambda} \cdot \frac{4r+\lambda}{5r-\lambda} \cdot \frac{6r-\lambda}{5r+\lambda} \cdot \text{etc.}$$

At

At vero in *Introductione in Analysin Infinitorum* pag. 147. ostendi esse

$$\text{tang.}\frac{m\pi}{2n} = \frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\frac{4n+m}{5n-m}\text{ etc.}$$

quae series manifesto in inuentam transformatur, statuendo $m=\lambda$ et $n=r$, ita vt nunc fit $P+Q=l\,\text{tang.}\frac{\lambda\pi}{2r}$, prorsus vti supra est inuentum.

Additamentum.

§. 33. In dissertatione Actorum Tomo V. parte I. inserta, vnde desumsi hoc theorema:

$$\int\frac{x^{k+n}}{1+x^{2k}}\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi}{2k\,\text{cos}\frac{n\pi}{2k}}$$

simul occurrunt sequentia:

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi}{k\,\text{cos}\frac{n\pi}{2k}}$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+x^{2k}}\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{2k\,\text{cos}\frac{n\pi}{2k}}$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{\pi}{k}\,\text{tang.}\frac{n\pi}{2k}$$

$$\int\frac{x^{k-n}-x^{k+n}}{1-x^{2k}}\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi}{2k}\,\text{tang.}\frac{n\pi}{2k}$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+2x^{k}\,\text{cos.}\eta+x^{2k}}\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=\infty\end{bmatrix}=\frac{2\pi\,\text{sin.}\frac{n\eta}{k}}{k\,\text{sin.}\eta\,\text{sin.}\frac{n\pi}{k}}$$

$$\int\frac{x^{k-n}+x^{k+n}}{1+2x^{k}\,\text{cos.}\eta+x^{2k}}\frac{dx}{x}\begin{bmatrix}\text{ab } x=0\\ \text{ad } x=1\end{bmatrix}=\frac{\pi\,\text{sin.}\frac{n\eta}{k}}{k\,\text{sin.}\eta\,\text{sin.}\frac{n\pi}{k}}$$

$$\int \frac{x^{k \pm n - 1}}{1 + 2x^k \cos.\eta + x^{2k}} \frac{dx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{2\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$$

quas formulas ergo ſimili modo tractare operae pretium erit.

§. 34. Incipiamus igitur a formula

$$\int \frac{x^{k-n} + x^{k+n}}{1 + x^{2k}} \frac{dx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = \frac{\pi}{2k \cos. \frac{n\pi}{2k}}$$

quia praecedens cum formula iam tractata prorſus conueniret, quae ſi ducatur in dn et ita integretur, vt integrale euaneſcat poſito $n = 0$, quoniam eſt

$$\int x^{k-n}\, dn = -\frac{x^{k-n} + x^k}{lx} \text{ et}$$

$$\int x^{k+n}\, dn = \frac{x^{k+n} - x^k}{lx},$$

tum vero, vt ante vidimus,

$$\int \frac{\pi\, dn}{2k} \cos. \frac{n\pi}{2k} = l \operatorname{tang}. \frac{(k+n)\pi}{4k},$$

prodibit haec integratio:

$$\int \frac{x^{k+n} - x^{k-n}}{(1 + x^{2k})} \frac{dx}{x\, lx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = 1 \end{bmatrix} = l \operatorname{tang}. \frac{(k+n)\pi}{4k},$$

qui ergo valor prorſus conuenit cum eo, quem pro formula

$$\int \frac{x^{k+n}}{1 + x^{2k}} \frac{dx}{x\, lx} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix}$$

inuenimus.

§. 35. Simili modo tractemus ſequentem formam:

$$\int \frac{x^{k-n} - x^{k+n}}{1 - x^{2k}} \frac{dx}{x} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{k} \operatorname{tang}. \frac{n\pi}{2k},$$

quae

quae, ducta in dn et vt supra integrata, praebet a parte sinistra

$$\int \frac{2x^k - x^{k-n} - x^{k+n}}{1 - x^{2k}} \frac{dx}{x\,l\,x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right],$$

a parte autem dextra

$$\int \frac{\pi\,dn}{k} \text{ tag. } \frac{n\pi}{2k} = \int \frac{\pi\,dn \text{ sin. } \frac{n\pi}{2k}}{k \text{ cos. } \frac{n\pi}{2k}}.$$

Ad hoc integrandum fiat $\frac{n\pi}{2k} = \phi$, eritque $\frac{\pi\,dn}{k} = 2\,d\,\phi$, sicque formula integranda erit

$$2\int \frac{d\Phi \text{ sin } \Phi}{\text{cos. } \Phi} = -2\,l \text{ cos. } \phi + C = 2\,l \text{ cos. } \frac{n\pi}{2k} + C.$$

Fiat igitur $n = 0$, esseque debebit $2\,l\,1 + C = 0$, ideoque constans $C = 0$, quocirca haec integratio nobis suppeditat sequentem formulam:

$$\int \frac{2x^k - x^{k-n} - x^{k+n}}{1 - x^{2k}} \frac{dx}{x\,l\,x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right] = 2\,l \text{ cos. } \frac{n\pi}{2k}$$

sequens autem formula $\left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = 1 \end{array}\right]$ singulari euolutione non indiget, cum eius valor sit huius semissis.

§. 36. Euoluamus casum quo $k = 2$ et $n = 1$, et ex parte sinistra habemus

$$-\int \frac{(1 - x)^2}{1 - x^4} \frac{dx}{l\,x} = -\int \frac{(1 - x)}{(1 + x)(1 + xx)} \frac{dx}{l\,x} \left[\begin{array}{l} \text{ab } x = 0 \\ \text{ad } x = \infty \end{array}\right];$$

at vero ex dextra parte: $2\,l \text{ cos. } \frac{\pi}{4} = -2\,l\,\sqrt{2} = -l\,2$. Verum fractio $\frac{1 - x}{(1 + x)(1 + xx)}$ resoluitur in has duas: $\frac{1}{1 + x} - \frac{x}{1 + xx}$, vnde formula nostra resoluitur in has duas:

$$\int \frac{dx}{(1 + x)\,l\,x} - \int \frac{x\,dx}{(1 + xx)\,l\,x} = l\,2.$$

 Sed

Sed ex forma generali

$$\int \frac{x^{\lambda-1}\,dx}{(1+x^r)\,lx} = l\,\text{tang.}\,\frac{\lambda\pi}{2r}$$

vtriusque formulae valor in infinitum excrescit, sicque nihil impedit, quo minus differentia $= l\,2$.

§. 37. Quod si hic in posteriore formula statuamus $xx = z$, ea abibit in hanc: $\int \frac{dz}{(1+z)\,lz}$, quae priori omnino est similis atque sub iisdem terminis integrationis continetur. Hic igitur iterum occurrit Paradoxon prorsus simile illi, quod ab Ill. *de la Grange* fuit memoratum: duae scilicet hic habentur formae prorsus pares: $\int \frac{dx}{(1+x)\,lx}$ et $\int \frac{dz}{(1+z)\,lz}$, quarum vtramque a termino o ad ∞ integrari oportet, nihilo tamen minus earum differentia non est nulla, sed vti vidimus $= l\,2$. Atque hinc Solutio huius Paradoxi in eo manifesto est sita, quod vtriusque integralis valor in infinitum excrescit.

§. 38. Quod si binas postremas formulas eodem modo tractare et per dn multiplicatas integrare velimus, a parte sinistra resultat ista formula integralis:

$$\int \frac{x^{k+n} - x^{k-n}}{(1+2x^k \cos.\eta + x^{2k})} \frac{dx}{x\,lx} \left[\begin{matrix}\text{ab } x = 0\\ \text{ad } x = \infty\end{matrix}\right];$$

pro dextra autem parte nanciscimur hanc formulam integralem:

$$\int \frac{2\pi\,dn\,\text{sin.}\frac{n\eta}{k}}{k\,\text{sin.}\,\eta\,\text{sin.}\frac{n\pi}{k}}$$

a termino $n = 0$ extendendam. Verum haec integratio nullo modo succedit; si enim ponamus $\frac{n\pi}{k} = \phi$, fiet $\frac{\eta n}{k} = \frac{\eta\phi}{\pi} = \alpha\phi$,

ponendo

ponendo $\frac{\eta}{\pi} = \alpha$, vnde formula integranda erit $\frac{\pi}{\sin.\eta} \int \frac{d\Phi \sin.\alpha\Phi}{\sin.\Phi}$, cuius valor aliter nisi per signum summatorium exprimi non potest, sicque nulla concinna Theoremata hinc deriuare licet.

§. 39. Quemadmodum autem hic, exponentem n vt variabilem spectando, transformationes per integrationem instituimus, ita etiam differentiatio egregias transformationes suppeditabit, quod argumentum vnica formula principali illustrasse sufficiet. Consideremus scilicet hanc formulam:

$$\int \frac{x^{k+n}}{1+x^{2k}} \frac{dx}{x} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{\pi}{2k \cos.\frac{n\pi}{2k}},$$

quae, sumto exponente n vt solo variabili, continuo differentietur, vbi notandum est esse $d.x^{k+n} = x^{k+n}\, dn\, lx$. At vero pro formula $\frac{\pi}{2k\cos.\frac{n\pi}{2k}}$ scribamus litteram v, quae ergo spectanda erit tanquam functio ipsius n, cuius ergo differentialia cuiusque ordinis sunt in nostra potestate. Hinc igitur sequentes reductiones consequemur:

$$\int \frac{x^{k+n}}{1+x^{2k}} \frac{dx}{x}\, lx = \frac{dv}{dn} \text{ siue}$$

$$\int \frac{x^{k+n-1}\, dx\, lx}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{dv}{dn}$$

$$\int \frac{x^{k+n-1}\, dx\, (lx)^2}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{ddv}{dn^2}$$

$$\int \frac{x^{k+n-1}\, dx\, (lx)^3}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^3v}{dn^3}$$

$$\int \frac{x^{k+n-1}\, dx\, (lx)^4}{1+x^{2k}} \begin{bmatrix} \text{ab } x=0 \\ \text{ad } x=\infty \end{bmatrix} = \frac{d^4v}{dn^4}$$

$$\int \frac{x^{k+\nu-1}\,dx\,(lx)^s}{1+x^{2k}}\left[\begin{array}{l}\text{ab } x=0\\ \text{ad } x=\infty\end{array}\right]=\frac{d^s \nu}{dn^s}.$$

§. 40. Cum igitur hinc totum negotium ad differentialia continua ipsius ν reducatur, ea sequenti modo commodissime reperire licebit. Cum enim sit

$$\nu=\frac{\pi}{2k\,\text{cos}.\frac{n\pi}{2k}},\ \text{erit}\ \nu\ \text{cos}.\frac{n\pi}{2k}=\frac{\pi}{2k},$$

hincque continuo differentiando obtinebimus sequentes formulas:

$$\frac{d\nu}{dn}\,\text{cos}.\frac{n\pi}{2k}-\frac{\pi}{2k}\,\nu\,\text{fin}.\frac{n\pi}{2k}=0$$

$$\frac{dd\nu}{dn^2}\,\text{cos}.\frac{n\pi}{2k}-\frac{2\pi}{2k}\,\frac{d\nu}{dn}\,\text{fin}.\frac{n\pi}{2k}-\frac{\pi\pi}{4kk}\,\nu\,\text{cos}.\frac{n\pi}{2k}=0$$

$$\frac{d^3\nu}{dn^3}\,\text{cos}.\frac{n\pi}{2k}-\frac{3\pi}{2k}\,\frac{dd\nu}{dn^2}\,\text{fin}.\frac{n\pi}{2k}-\frac{3\pi\pi}{4kk}\,\frac{d\nu}{dn}\,\text{cos}.\frac{n\pi}{2k}+\frac{\pi^3}{8k^3}\,\nu\,\text{fin}.\frac{n\pi}{2k}=0$$

$$\frac{d^4\nu}{dn^4}\,\text{cos}.\frac{n\pi}{2k}-\frac{4\pi}{2k}\,\frac{d^3\nu}{dn^3}\,\text{fin}.\frac{n\pi}{2k}-\frac{6\pi\pi}{4kk}\,\frac{dd\nu}{dn^2}\,\text{cos}.\frac{n\pi}{2k}+\frac{4\pi^3}{8k^3}\,\frac{d\nu}{dn}\,\text{fin}.\frac{n\pi}{2k}$$
$$+\frac{\pi^4}{16k^4}\,\nu\,\text{cos}.\frac{n\pi}{2k}=0$$

etc. etc.

vnde singula differentialia altiora ex inferioribus formari possunt.

§. 41. Quo autem hae operationes magis subleuentur, statuamus breuitatis gratia $\frac{\pi}{2k}=\alpha$, vt sit $\nu=\frac{\alpha}{\text{cof}.\,\alpha n}$, atque singula differentialia ex superioribus aequationibus sequenti modo determinabuntur:

$$\frac{d\nu}{dn}=\alpha\,\nu\ \text{tang}.\,\alpha n$$

$$\frac{dd\nu}{dn^2}=2\alpha\,\frac{d\nu}{dn}\,\text{tang}.\,\alpha n+\alpha\alpha\,\nu$$

$$\frac{d^3\nu}{dn^3}=3\alpha\,\frac{dd\nu}{dn^2}\,\text{tang}.\,\alpha n+3\alpha\alpha\,\frac{d\nu}{dn}-\alpha^3\,\nu\ \text{tang}.\,\alpha n$$

$$\frac{d^4 v}{d n^4} = 4\alpha \frac{d^3 v}{d n^3} \text{tang.}\, \alpha n + 6\alpha\alpha \frac{dd v}{d n^2} + 4\alpha^3 \frac{d v}{d n} \text{tang.}\, \alpha n - \alpha^4 v$$

$$\frac{d^5 v}{d n^5} = 5\alpha \frac{d^4 v}{d n^4} \text{tang.}\, \alpha n + 10\alpha\alpha \frac{d^3 v}{d n^3} - 10\alpha^3 \frac{dd v}{d n^2} \text{tang.}\, \alpha n - 5\alpha^4 \frac{d v}{d n}$$
$$+ \alpha^5 v \,\text{tang.}\, \alpha n$$

etc. etc.

Quod si breuitatis gratia insuper statuamus $\text{tang.}\, \alpha n = t$, et praecedentes valores in sequentibus substituamus, reperiemus:

$$\frac{d v}{d n} = \alpha v . t$$

$$\frac{dd v}{d n^2} = \alpha\alpha v (2tt + 1)$$

$$\frac{d^3 v}{d n^3} = \alpha^3 v (6t^3 + 5t)$$

$$\frac{d^4 v}{d n^4} = \alpha^4 v (24t^4 + 28tt + 5)$$

$$\frac{d^5 v}{d n^5} = \alpha^5 v (120t^5 + 180t^3 + 61t)$$

$$\frac{d^6 v}{d n^6} = \alpha^6 v (720t^6 + 1320t^4 + 662tt + 61)$$

§. 42. Ex consideratione harum expressionum facilis erui potest operatio, cuius ope ex qualibet earum expressionum sequens colligi potest. Sit enim pro differentiali ordinis indefiniti

$$\frac{d^\lambda v}{d n^\lambda} = \alpha^\lambda v . P$$

at pro ordine sequente

$$\frac{d^{\lambda+1} v}{d n^{\lambda+1}} = \alpha^{\lambda+1} v . Q$$

et quoniam vidimus valorem ipsius P talem habere formam:

$$P = A t^\lambda + B t^{\lambda-2} + C t^{\lambda-4} + D t^{\lambda-6};$$

rum

tum valor ipsius Q ex sequentibus binis seriebus erit compositus:

$$Q = (\lambda+1) A t^{\lambda+1} + (\lambda-1) B t^{\lambda-1} + (\lambda-3) C t^{\lambda-3} + (\lambda-5) D t^{\lambda-5} \text{ etc.}$$
$$+ \lambda A t^{\lambda-1} + (\lambda-2) B t^{\lambda-3} + (\lambda-4) C t^{\lambda-5} + \text{ etc.}$$

vnde patet hanc determinationem ita repraesentari posse, vt sit

$$Q = \frac{t.d.Pt}{dt} + \frac{dP}{dt}.$$

§. 43. Haec vero formula, qua ex cognito valore P sequens Q deriuatur, etiam ex ipsa natura rei sequenti modo ostendi potest. Cum per hypothesin sit

$$\frac{d^\lambda v}{dn^\lambda} = \alpha^\lambda v P,$$

erit differentiando

$$\frac{d^{\lambda+1} v}{dn^\lambda} = \alpha^\lambda P dv + \alpha^\lambda v dP;$$

initio autem vidimus esse $\frac{dv}{dn} = \alpha v.t$, siue $dv = \alpha v t.dn$ quo valore substituto fit

$$\frac{d^{\lambda+1} v}{dn^{\lambda+1}} = \alpha^{\lambda+1} v P t + \alpha^\lambda v \frac{dP}{dn};$$

tum vero assumsimus $t = \text{tang.} \alpha n$, vnde differentiando fit $\alpha dn = \frac{dt}{1+tt}$, quo valore in postremo termino substituto obtinebitur

$$\frac{d^{\lambda+1} v}{dn^{\lambda+1}} = \alpha^{\lambda+1} v P t + \alpha^{\lambda+1} v \frac{dP(1+tt)}{dt} = \alpha^{\lambda+1} v \left(Pt + \frac{dP(1+tt)}{dt}\right)$$

quae forma manifesto reducitur ad hanc:

$$\frac{d^{\lambda+1} v}{dn^{\lambda+1}} = \alpha^{\lambda+1} v. \frac{t.d.Pt + dP}{dt},$$

ita

ita vt sit

$$Q = \frac{t.d.Pt + dP}{dt} = Pt + \frac{dP(1+tt)}{dt};$$

vnde intelligitur, si sumatur $tt + 1 = 0$; quo facto in nostris formulis signa terminorum alternabuntur, et omissa litera t, fieri $Q = P$; vnde patet, hoc casu omnes formulas superiores eundem valorem esse adepturas, id quod etiam ex formulis supra exhibitis manifestum est, ex quibus erit $2 - 1 = 1$; $6 - 5 = 1$; $24 - 28 + 5 = 1$; $120 - 180 + 61 = 1$; $720 - 1320 + 662 - 61 = 1$; etc. vnde insigne criterium obtinetur, vtrum formulae istae recte sint per calculum definitae.

INVESTIGATIO FORMVLAE INTEGRALIS

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^n}$$

CASV QVO POST INTEGRATIONEM STATVITVR

$$x = \infty.$$

§. 1.

Iam satis notum est, huius formulae integrale partim logarithmos, partim arcus circulares complecti, et partes logarithmicas hanc progressionem constituere:

$$-\tfrac{2}{k}\operatorname{cos.}\tfrac{m\pi}{k}\, l\sqrt{(1-2x\operatorname{cos.}\tfrac{\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\operatorname{cos.}\tfrac{3m\pi}{k}\, l\sqrt{(1-2x\operatorname{cos.}\tfrac{3\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\operatorname{cos.}\tfrac{5m\pi}{k}\, l\sqrt{(1-2x\operatorname{cos.}\tfrac{5\pi}{k}+xx)}$$
$$-\tfrac{2}{k}\operatorname{cos.}\tfrac{7m\pi}{k}\, l\sqrt{(1-2x\operatorname{cos.}\tfrac{7\pi}{k}+xx)}$$
$$- - - - - - - - - - -$$
$$- - - - - - - - - - -$$
$$-\tfrac{2}{k}\operatorname{cos.}\tfrac{im\pi}{k}\, l\sqrt{(1-2x\operatorname{cos.}\tfrac{i\pi}{k}+xx)}$$

vbi i denotat numerum imparem non maiorem quam k. Hinc si k fuerit numerus par, erit $i=k-1$; ac si k fuerit numerus impar, hanc progressionem continuari oportet vsque ad $i=k$; eius vero coefficiens duplo minor capi debet, seu loco $-\frac{2}{k}$ tantum scribi debet $-\frac{1}{k}$, cuius irregularitatis ratio in *Calculo Integrali* est exposita.

§. 2.

§. 2. Cum hae partes sponte iam euanescant posito $x = 0$, statuamus statim $x = \infty$, et cum in genere sit

$$\sqrt{(1 - 2x \operatorname{cos}. \omega + xx)} = x - \operatorname{cos}. \omega, \text{ erit}$$

$$l\sqrt{(1 - 2x\operatorname{cos}.\omega + xx)} = l(x - \operatorname{cos}.\omega) = lx - \tfrac{\operatorname{cos}.\omega}{x} = lx, \text{ ob } \tfrac{\operatorname{cos}.\omega}{x} = 0;$$

omnes ergo illi logarithmi reducuntur ad eandem formam lx, quae multiplicanda est per hanc seriem:

$$-\tfrac{2}{k}\operatorname{cos}.\tfrac{m\pi}{k} - \tfrac{2}{k}\operatorname{cos}.\tfrac{3m\pi}{k} - \tfrac{2}{k}\operatorname{cos}.\tfrac{5m\pi}{k} - \cdots - \tfrac{2}{k}\operatorname{cos}.\tfrac{im\pi}{k},$$

vbi, vt diximus, i denotat maximum numerum imparem ipso k non maiorem, hac tamen restrictione, vt, si k fuerit impar, ideoque $i = k$, vltimum membrum ad dimidium reduci debeat. Quamobrem, si huius progressionis summam inuestigare velimus, duo casus erunt constituendi: alter quo k est numerus par et $i = k - 1$, alter vero quo k est impar et $i = k$.

Euolutio casus prioris, quo k est numerus par et $i = k - 1$.

§. 3. Hoc ergo casu, posito $x = \infty$, formula $-\frac{2}{klx}$ multiplicatur per hanc Cosinuum seriem:

$$\operatorname{cos}.\tfrac{m\pi}{k} + \operatorname{cos}.\tfrac{3m\pi}{k} + \operatorname{cos}.\tfrac{5m\pi}{k} + \operatorname{cos}.\tfrac{7m\pi}{k} + \cdots + \operatorname{cos}.\tfrac{(k-1)m\pi}{k},$$

cuius summam statuamus $= S$. Ducamus hanc seriem in $\operatorname{sin}.\frac{m\pi}{k}$, et cum in genere sit

$$\operatorname{sin}.\tfrac{m\pi}{k}\operatorname{cos}.\tfrac{im\pi}{k} = \tfrac{1}{2}\operatorname{sin}.\tfrac{(i+1)m\pi}{k} - \tfrac{1}{2}\operatorname{sin}.\tfrac{(i-1)m\pi}{k},$$

facta hac reductione habebimus

$$S\operatorname{sin}.\tfrac{m\pi}{k} = \tfrac{1}{2}\operatorname{sin}.\tfrac{2m\pi}{k} + \tfrac{1}{2}\operatorname{sin}.\tfrac{4m\pi}{k} + \tfrac{1}{2}\operatorname{sin}.\tfrac{6m\pi}{k} \cdots + \tfrac{1}{2}\operatorname{sin}.\tfrac{(k-1)m\pi}{k}$$
$$- \tfrac{1}{2}\operatorname{sin}.\tfrac{2m\pi}{k} - \tfrac{1}{2}\operatorname{sin}.\tfrac{4m\pi}{k} - \tfrac{1}{2}\operatorname{sin}.\tfrac{6m\pi}{k} - \cdots - \tfrac{1}{2}\operatorname{sin}.\tfrac{(k-2)m\pi}{k};$$

 vbi

vbi omnes termini praeter vltimum manifesto se destruunt, ita vt sit

$$S\,\text{sin.}\,\frac{m\pi}{k} = \tfrac{1}{2}\,\text{sin.}\,m\pi.$$

Iam vero quia nostri coefficientes m et k supponuntur integri, vtique erit $\text{sin.}\,m\pi = 0$, ideoque etiam $S = 0$, nisi forte etiam fuerit $\text{sin.}\,\frac{m\pi}{k} = 0$, qui autem casus locum habere nequit, quoniam in integratione formulae propositae $\frac{x^{m-1}\,dx}{1+x^k}$ semper assumi solet esse $m < k$. Hoc igitur modo euictum est, casu quo post integrationem statuitur $x=\infty$, omnes partes logarithmicas integralis se destruere.

Euolutio casus alterius, quo est k numerus impar et $i = k$.

§. 4. Hoc ergo casu, sumto $x = \infty$, formula lx multiplicatur per hanc seriem:

$$-\tfrac{2}{k}\,\text{cos.}\,\frac{m\pi}{k} - \tfrac{2}{k}\,\text{cos.}\,\frac{3m\pi}{k} - \tfrac{2}{k}\,\text{cos.}\,\frac{5m\pi}{k} - - - - - - \tfrac{1}{k}\,\text{cos.}\,\frac{im\pi}{k},$$

vbi terminus penultimus est $-\frac{2}{k}\,\text{cos.}\,\frac{(k-2)m\pi}{k}$, pro vltimo vero termino erit $\text{cos.}\,m\pi = \pm 1$, signo superiore valente si n sit numerus par, inferiore si impar; quare remoto termino vltimo pro reliquis ponamus

$$\text{cos.}\,\frac{m\pi}{k} + \text{cos.}\,\frac{3m\pi}{k} + \text{cos.}\,\frac{5m\pi}{k} + - - - - - + \text{cos.}\,\frac{(k-2)m\pi}{k} = S$$

ita vt multiplicator ipsius logarithmi x sit

$$-\frac{2S}{k} - \tfrac{1}{k}\,\text{cos.}\,m\pi.$$

Hinc procedendo vt ante fiet

$$S\,\text{sin.}\,\frac{m\pi}{k} = \tfrac{1}{2}\,\text{sin.}\,\frac{2m\pi}{k} + \tfrac{1}{2}\,\text{sin.}\,\frac{4m\pi}{k} + \tfrac{1}{2}\,\text{sin.}\,\frac{6m\pi}{k} - - - - - + \tfrac{1}{2}\,\text{sin.}\,\frac{(k-1)m\pi}{k}$$
$$-\tfrac{1}{2}\,\text{sin.}\,\frac{2m\pi}{k} - \tfrac{1}{2}\,\text{sin.}\,\frac{4m\pi}{k} - \tfrac{1}{2}\,\text{sin.}\,\frac{6m\pi}{k} - - - - - - \tfrac{1}{2}\,\text{sin.}\,\frac{(k-3)m\pi}{k};$$

vbi

vbi iterum omnes termini praeter vltimum se mutuo tollunt, ita vt hinc prodeat

$$S \sin.\frac{m\pi}{k} = \frac{1}{2}\sin.\frac{(k-1)m\pi}{k} = \frac{1}{2}\sin.\left(m\pi - \frac{m\pi}{k}\right);$$

at vero est

$$\sin.\left(m\pi - \frac{m\pi}{k}\right) = \sin.m\pi\cos.\frac{m\pi}{k} - \cos.m\pi\sin.\frac{m\pi}{k},$$

vbi notetur esse $\sin.m\pi = 0$, ob m numerum integrum; habebimus ergo

$$S\sin.\frac{m\pi}{k} = -\frac{1}{2}\cos.m\pi\sin.\frac{m\pi}{k}, \text{ siue } S = -\frac{1}{2}\cos.m\pi,$$

consequenter multiplicator ipsius lx erit

$$= \frac{1}{k}\cos.m\pi - \frac{1}{k}\cos.m\pi = 0,$$

sicque manifestum est, siue k sit numerus par siue impar, omnia membra logarithmica in nostro integrali se mutuo destruere, siquidem post integrationem statuamus $x = \infty$, quemadmodum hic semper supponimus.

§. 5. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulae componitur. Hae autem partes sequentem progressionem constituere sunt compertae:

$$\frac{2}{k}\sin.\frac{m\pi}{k}\text{A tang.}\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}} + \frac{2}{k}\sin.\frac{3m\pi}{k}\text{A tang.}\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}$$

$$+\frac{2}{k}\sin.\frac{5m\pi}{k}\text{A tang.}\frac{x\sin.\frac{5\pi}{k}}{1-x\cos.\frac{5\pi}{k}} + \frac{2}{k}\sin.\frac{7m\pi}{k}\text{A tang.}\frac{x\sin.\frac{5\pi}{k}}{1-x\cos.\frac{5\pi}{k}}$$

$$+ - - - - - - - - - - + \frac{2}{k}\sin.\frac{im\pi}{k}\text{A tang.}\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}$$

vbi in vltimo membro est vel $i = k-1$, vel $i = k$; prius scilicet valet si i est numerus par, posterius si impar.

§. 6. Cum etiam omnia haec membra euanescant posito $x = 0$, faciamus pro instituto nostro $x = \infty$. In genere igitur fiet

$$\text{A tang.} \frac{x \sin. \frac{i\pi}{k}}{1 - x \cos. \frac{i\pi}{k}} = \text{A tang.} \left(- \text{tang.} \frac{i\pi}{k}\right).$$

Est vero

$$- \text{tang.} \frac{i\pi}{k} = + \text{tang.} \frac{(k-i)\pi}{k},$$

ex quo hic arcus fit $= \frac{(k-i)\pi}{k}$. Hinc ergo loco i scribendo successiue numeros 1, 3, 5, 7 etc. istae partes nostri integralis quaesiti erunt

$$\frac{2(k-1)\pi}{kk} \sin. \frac{m\pi}{k} + \frac{2(k-3)\pi}{kk} \sin. \frac{3m\pi}{k} + \frac{2(k-5)\pi}{kk} \sin. \frac{5m\pi}{k}$$
$$+ \frac{2(k-7)\pi}{kk} \sin. \frac{7m\pi}{k} + \frac{2(k-9)\pi}{kk} \sin. \frac{9m\pi}{k} + \dots \frac{2(k-i)\pi}{kk} \sin. \frac{im\pi}{k}$$

vbi casu, quo k est numerus par, progredi oportet vsque ad $i = k - 1$; ac si k sit numerus impar, vsque ad $i = k$.

§. 7. Statuamus breuitatis gratia

$$(k-1) \sin. \frac{m\pi}{k} + (k-3) \sin. \frac{3m\pi}{k} + (k-5) \sin. \frac{5m\pi}{k} + \dots$$
$$+ (k-i) \sin. \frac{im\pi}{k} = S$$

ita vt integrale quaesitum sit $\frac{2\pi S}{kk}$, quandoquidem partes logarithmicae se mutuo destruxerunt. Multiplicemus nunc vtrinque per $2 \sin. \frac{m\pi}{k}$, et cum in genere sit

$$2 \sin. \frac{m\pi}{k} \sin. \frac{im\pi}{k} = \cos. \frac{(i-1)m\pi}{k} - \cos. \frac{(i+1)m\pi}{k},$$

facta substitutione erit

$$2\,S \sin.\frac{m\pi}{k} = (k-1)\cos.\frac{0\,m\pi}{k} + (k-3)\cos.\frac{2\,m\pi}{k} + (k-5)\cos.\frac{4\,m\pi}{k} - - - -$$
$$-(k-1)\cos.\frac{2\,m\pi}{k} - (k-3)\cos.\frac{4\,m\pi}{k} - (k-5)\cos.\frac{6\,m\pi}{k} - - - -$$
$$- - - - + (k-i)\cos.\frac{(i-1)m\pi}{k}$$
$$-(k-i)\cos.\frac{(i+1)m\pi}{k}$$

quae series manifesto contrahitur in sequentem:

$$2\,S\sin.\frac{m\pi}{k} = (k-1) - 2\cos.\frac{2\,m\pi}{k} - 2\cos.\frac{4\,m\pi}{k} - 2\cos.\frac{6\,m\pi}{k} - - - - 2\cos.\frac{(i-1)m\pi}{k}$$
$$-(k-i)\cos.\frac{(i+1)m\pi}{k}$$

vbi, primo et vltimo membro sublatis, regularem termini intermedii constituunt seriem, pro cuius valore inuestigando ponamus

$$T = \cos.\frac{2\,m\pi}{k} + \cos.\frac{4\,m\pi}{k} + \cos.\frac{6\,m\pi}{k} + - - - - - + \cos.\frac{(i-1)m\pi}{k},$$

ita vt sit

$$2\,S \sin.\frac{m\pi}{k} = k - 1 - 2\,T - (k-i)\cos.\frac{(i+1)m\pi}{k}.$$

Hic autem iterum conuenit duos casus perpendere, prout k fuerit par vel impar.

Euolutio casus prioris, quo k est numerus par et $i = k-1$.

§. 8. Hoc ergo casus habebimus

$$T = \cos.\frac{2\,m\pi}{k} + \cos.\frac{4\,m\pi}{k} + \cos.\frac{6\,m\pi}{k} + - - - - - \cos.\frac{(k-2)m\pi}{k}.$$

Multiplicemus denuo per $2 \sin.\frac{m\pi}{k}$, et per reductiones supra indicatas habebimus

$$2\,T \sin.\frac{m\pi}{k} = \sin.\frac{3\,m\pi}{k} + \sin.\frac{5\,m\pi}{k} + \sin.\frac{7\,m\pi}{k} + - - - - - \sin.\frac{(k-1)m\pi}{k}$$
$$-\sin.\frac{m\pi}{k} - \sin.\frac{3\,m\pi}{k} - \sin.\frac{5\,m\pi}{k} - - - - - - \sin.\frac{(k-3)m\pi}{k};$$

deletis igitur terminis se mutuo tollentibus erit

$$2\,T\,\text{fin.}\,\frac{m\pi}{k} = -\,\text{fin.}\,\frac{m\pi}{k} + \text{fin.}\,\frac{(k-1)m\pi}{k}.$$

Eſt vero

$$\text{fin.}\,\frac{(k-1)m\pi}{k} = \text{fin.}\left(m\pi - \frac{m\pi}{k}\right) = \text{fin.}\,m\pi\,\text{cof.}\,\frac{m\pi}{k} - \text{cof.}\,m\pi\,\text{fin.}\,\frac{m\pi}{k},$$

vbi $\text{fin.}\,m\pi = 0$, quamobrem fiet $2\,T = -1 - \text{cof.}\,m\pi$.

§. 9. Inuento valore pro T colligitur fore

$$2\,S\,\text{fin.}\,\frac{m\pi}{k} = k, \text{ ideoque } S = \frac{k}{2\,\text{fin.}\,\frac{m\pi}{k}}.$$

Denique vero ipſe valor formulae noſtrae integralis, quem quaerimus, erit $\frac{2\pi Si}{kk}$, et nunc manifeſtum eſt, integrale noſtrae formulae, caſu quo S eſt numerus par, fore $\frac{\pi}{k.\text{fin.}\,\frac{m\pi}{k}}$, ſiquidem poſt integrationem ſtatuatur $x = \infty$.

Euolutio alterius caſus, quo k eſt numerus impar et $i = k$.

§. 10. Hoc ergo caſu eſt

$$T = \text{cof.}\,\frac{2m\pi}{k} + \text{cof.}\,\frac{4m\pi}{k} + \text{cof.}\,\frac{6m\pi}{k} + \cdots + \text{cof.}\,\frac{(k-1)m\pi}{k},$$

quae ſeries multiplicata per $2\,\text{fin.}\,\frac{m\pi}{k}$ producet vt ante

$$2\,T\,\text{fin.}\,\frac{m\pi}{k} = \text{fin.}\,\frac{3m\pi}{k} + \text{fin.}\,\frac{5m\pi}{k} + \text{fin.}\,\frac{7m\pi}{k} + \cdots + \text{fin.}\,\frac{km\pi}{k}$$
$$-\,\text{fin.}\,\frac{m\pi}{k} - \text{fin.}\,\frac{3m\pi}{k} - \text{fin.}\,\frac{5m\pi}{k} - \cdots - \text{fin.}\,\frac{(k-2)m\pi}{k},$$

vnde deletis terminis ſe mutuo tollentibus reperietur

$$2\,T\,\text{fin.}\,\frac{m\pi}{k} = -\,\text{fin.}\,\frac{m\pi}{k} + \text{fin.}\,m\pi$$

ideoque

$$2\,T = -1 + \frac{\text{fin.}\,m\pi}{\text{fin.}\,\frac{m\pi}{k}} = 1, \text{ ob fin.}\,m\pi = 0,$$

hincque

hincque porro fiet

$$2\,S\,\text{fin.}\,\frac{m\pi}{k} = k;$$

quare cum valor integralis quaesitus sit $\frac{2\pi S}{kk}$, erit etiam hoc casu integrale nostrum $= \frac{\pi}{k\,\text{fin.}\,\frac{m\pi}{k}}$, prorsus vti praecedente casu. Hinc ergo deducimus sequens

Theorema.

§. 11. *Si haec formula differentialis:* $\frac{x^{m-1}dx}{1+x^k}$, *ita integretur, vt, posito* $x = 0$, *integrale euanescat, tum vero statuatur* $x = \infty$, *valor inde resultans semper erit* $\frac{\pi}{k\,\text{fin.}\,\frac{m\pi}{k}}$, *siue* k *sit numerus par, siue impar.* Huius Theorematis demonstratio ex praecedentibus est manifesta.

§. 12. In euolutione huius formulae assumsimus esse $m < k$, quia alioquin membra logarithmica se non destruissent; at vero ne hac quidem limitatione nunc amplius est opus. Casu enim quo foret $m = k$, integrale formulae $\frac{x^{m-1}dx}{1+x^k}$ esset $\frac{1}{k}l(1+x^k)$, quod facto $x = \infty$ fieret etiam ∞; verum hoc idem indicat, nostrum integrale esse $\frac{\pi}{k\,\text{fin}\,\pi} = \infty$. Dummodo ergo m non fuerit maius quam k, nostra formula veritati semper est consentanea.

§. 13. Quin etiam ne quidem necesse est vt exponentes m et k sint numeri integri, dummodo non fuerit $m > k$; si enim fuerit $m = \frac{\mu}{\lambda}$ et $k = \frac{\nu}{\lambda}$, erit valor per no-

stram

ſtram formulam $\frac{\pi\lambda}{\varkappa \sin. \frac{\mu\pi}{\varkappa}}$, cuius veritas ita oſtenditur. Quia hoc caſu formula integranda eſt $\int \frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\varkappa}{\lambda}}} \frac{dx}{x}$, ſtatuatur $x = y^{\lambda}$, erit $\frac{dx}{x} = \frac{\lambda dy}{y}$ et formula fiet

$$\int \frac{y^{\mu}}{1+y^{\varkappa}} \frac{\lambda dy}{y} = \lambda \int \frac{y^{\mu-1} dy}{1+y^{\varkappa}}$$

cuius valor vtique erit $\frac{\lambda\pi}{\lambda \sin. \frac{\mu\pi}{\varkappa}}$.

Alia demonſtratio Theorematis.

§. 14. Denotet P valorem integralis $\int \frac{x^m}{1+x^k} \frac{dx}{x}$ a termino $x = 0$ vsque ad $x = 1$; at Q valorem eiusdem integralis a termino $x = 1$ vsque ad $x = \infty$, ita vt P + Q praebeat eum ipſum valorem, qui in theoremate continetur. Nunc pro valore Q inueniendo ſtatuatur $x = \frac{1}{y}$, vnde fit $\frac{dx}{x} = -\frac{dy}{y}$, fietque

$$Q = \int \frac{y^{-m}}{1+y^{-k}} \frac{-dy}{y} = -\int \frac{y^{k-m}}{1+y^{k}} \frac{dy}{y}$$

a termino $y = 1$ vsque ad $y = 0$. Hinc igitur commutatis terminis erit $Q = +\int \frac{y^{k-m}}{1+y^k} \frac{dy}{y}$, a termino $y = 0$ vsque ad $y = 1$. Iam quia hoc integrali expedito littera y ex calculo egreditur, loco y ſcribere licebit x, ita vt ſit

$$Q = \int \frac{x^{k-m}}{1+x^k} \frac{dx}{x},$$

quo

quo facto habebimus

$$P + Q = \int \frac{x^m + x^{k-m}}{1 + x^k} \cdot \frac{dx}{x}$$

a termino $x = 0$ vsque ad terminum $x = 1$. Verum non ita pridem demonstraui, valorem huius formulae integralis intra terminos $x = 0$ et $x = 1$ contentum esse $= \frac{\pi}{k \sin. \frac{m\pi}{k}}$. Hinc igitur nascitur sequens Theorema non minus notatu dignum.

Theorema.

§. 15. *Valor huius formulae integralis:*

$$\int \frac{x^m + x^{k-m}}{1 + x^k} \cdot \frac{dx}{x}$$

intra terminos $x = 0$ *et* $x = 1$ *contentus, aequalis est valori istius integralis:* $\int \frac{x^m}{1 + x^k} \cdot \frac{dx}{x}$, *intra terminos* $x = 0$ *et* $x = \infty$ *contento.*

§. 16. His expensis formulam integralem in titulo propositam aggrediamur, et quo eam ad formam hactenus tractatam reducamus, in subsidium vocemus sequentem reductionem:

$$\int \frac{x^{m-1}\, dx}{(1+x^k)^{\lambda+1}} = \frac{A x^m}{(1+x^k)^\lambda} + B \int \frac{x^{m-1}\, dx}{(1+x^k)^\lambda},$$

vnde facta differentiatione prodit sequens aequatio:

$$\frac{x^{m-1}\, dx}{(1+x^k)^{\lambda+1}} = \frac{m A x^{m-1}\, dx}{(1+x^k)^\lambda} - \frac{\lambda k A x^{m+k-1}\, dx}{(1+x^k)^{\lambda+1}} + \frac{B x^{m-1}\, dx}{(1+x^k)^\lambda},$$

 quae

quae aequatio, per $x^{m-1}dx$ diuisa ac per $(1+x^k)^\lambda$ multiplicata, terminum negatiuum a dextra ad sinistram transponendo, erit

$$\frac{1+\lambda k A x^k}{1+x^k} = mA + B$$

quae aequatio manifesto subsistere nequit, nisi sit $\lambda k A = 1$, siue $A = \frac{1}{\lambda k}$, vnde erit $1 = mA + B = \frac{m}{\lambda k} + B$, sicque erit $B = 1 - \frac{m}{\lambda k}$.

§. 17. Inuentis his valoribus pro litteris A et B, primum assumimus, integralia ita capi, vt euanescant posito $x = 0$; tum vero posito $x = \infty$, quia exponens n minor supponitur quam k, membrum absolutum littera A affectum sponte euanescit, ita vt hoc casu $x = \infty$ fiat

$$\int \frac{x^{m-1}dx}{(1+x^k)^{\lambda+1}} = \left(1 - \frac{m}{\lambda k}\right)\int \frac{x^{m-1}dx}{(1+x^k)^\lambda}.$$

Quod si iam primo capiamus $\lambda = 1$, quia ante inuenimus pro eodem casu $x = \infty$ esse

$$\int \frac{x^{m-1}dx}{1+x^k} = \frac{\pi}{k \sin. \frac{m\pi}{k}},$$

habebimus valorem istius integralis

$$\int \frac{x^{m-1}dx}{(1+x^k)^2} = \left(1 - \frac{m}{k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}};$$

si quidem integrale etiam a termino $x = 0$ vsque ad terminum $x = \infty$ extendatur.

§. 18. Quod si iam simili modo ponamus $\lambda = 2$, reperietur pro iisdem terminis integrationis

$\int x$

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^3} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\frac{\pi}{k\,\text{sin.}\,\frac{m\pi}{k}};$$

eodem modo si litterae λ continuo maiores valores tribuantur, reperientur sequentes integralium formae omni attentione dignae:

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^4} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\frac{\pi}{k\,\text{sin.}\,\frac{m\pi}{k}}$$

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^5} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\frac{\pi}{k\,\text{sin.}\,\frac{m\pi}{k}}$$

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^6} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\left(1-\frac{m}{5k}\right)\frac{\pi}{k\,\text{sin.}\,\frac{m\pi}{k}}$$

etc. etc.

§. 19. Quare si littera n denotet numerum quemcunque integrum, pro formula in titulo expressa, si eius integrale a termino $x=0$ vsque ad $x=\infty$ extendatur, eius valor sequenti modo se habebit:

$$\left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\ldots\left(1-\frac{m}{(n-1)k}\right)\frac{\pi}{k\,\text{sin.}\,\frac{m\pi}{k}}$$

qui ergo conueniet huic formulae integrali:

$$\int \frac{x^{m-1}\,dx}{(1+x^k)^n}.$$

§. 20. Hic quidem necessario pro n alii numeri praeter integros accipi non licet: at vero per methodum interpolationum, quae fusius iam passim est explicata, hanc integrationem etiam ad casus, quibus exponens n est numerus fractus, extendere licet. Quod si enim sequentes formulae integrales a termino $y=0$ vsque ad $y=1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit:

$$\int \frac{x^{m-1}dx}{(1+x^k)^n} = \frac{\pi}{k\sin.\frac{m\pi}{k}} \cdot \frac{\int y^{nk-m-1}dy(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}dy(1-y^k)^{\frac{m}{k}-1}}.$$

Vnde si fuerit $m=1$ et $k=2$, sequitur fore

$$\int \frac{dx}{(1+xx)^n} = \frac{\pi}{2}\int \frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}} : \int \frac{dy}{\sqrt{(1-yy)}} = \int \frac{y^{2(n-1)}dy}{\sqrt{(1-yy)}}$$

Ita si $n=\frac{3}{2}$ erit

$$\int \frac{dx}{(1+xx)^{\frac{3}{2}}} = \int \frac{ydy}{\sqrt{(1-yy)}}$$

cuius veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{1+xx}}$, posterius vero $=1-\sqrt{(1-yy)}$, quae, facto $x=\infty$ et $y=1$, vtique fiunt aequalia. Caeterum pro hac integratione generali notasse iuuabit, exponentem vnitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

IN-

INVESTIGATIO VALORIS INTEGRALIS

$$\int \frac{x^{m-1}\, dx}{1 - 2x^k \cos. \theta + x^{2k}}$$

A TERMINO $x = 0$ VSQVE AD $x = \infty$ EXTENSI.

§. 1.

Quaeramus primo integrale formulae propositae indefinitum, atque adeo omnes operationes ex primis Analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolui nequit, sit in genere eius factor duplicatus quicunque $1 - 2x\cos. \omega + xx$; euidens enim est, denominatorem fore productum ex k huiusmodi factoribus duplicatis. Cum igitur, posito hoc factore $= 0$, fiat $x = \cos. \omega \pm \sqrt{-1} \sin. \omega$, etiam ipse denominator duplici modo euanescere debebit, siue si ponatur

$$x = \cos. \omega + \sqrt{-1} \sin. \omega, \text{ siue}$$
$$x = \cos. \omega - \sqrt{-1} \sin. \omega.$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, vt sit

$$(\cos. \omega \pm \sqrt{-1} \sin.)^\lambda = \cos. \lambda\omega \pm \sqrt{-1} \sin. \lambda\omega;$$

hinc igitur erit

$$x^k = \cos. k\omega \pm \sqrt{-1} \sin. k\omega \text{ et}$$
$$x^{2k} = \cos. 2k\omega \pm \sqrt{-1} \sin. 2k\omega.$$

Sub-

Subſtituamus ergo hos valores et denominator noſter euadet

$$1 - 2 \cos.\theta \cos. k\omega + \cos. 2k\omega =$$
$$\pm 2\sqrt{-1} \cos.\theta \sin. k\omega \mp \sqrt{-1} \sin. 2k\omega$$

§. 2. Perſpicuum igitur eſt huius aequationis tam terminos reales quam imaginarios ſeorſim ſe mutuo tollere debere, vnde naſcuntur hae duae aequationes:

$$\text{I.}\quad 1 - 2\cos.\theta \cos. k\omega + \cos. 2k\omega = 0$$

$$\text{II.}\quad - 2\cos.\theta \sin. k\omega + \sin. 2k\omega = 0.$$

Cum igitur ſit

$$\sin. 2k\omega = 2\sin. k\omega \cos. k\omega,$$

poſterior aequatio induet hanc formam:

$$- 2\cos.\theta \sin. k\omega + 2\sin. k\omega \cos. k\omega,$$

quae per $2\sin. k\omega$ diuiſa dat $+\cos. k\omega = \cos.\theta$, ideoque

$$\cos. 2k\omega = \cos. 2\theta = \cos.\theta^2 - \sin.\theta^2 = 2\cos.\theta^2 - 1,$$

qui valores in aequatione priore ſubſtitui praebent aequationem identicam, ita vt vtrique aequationi ſatisfiat ſumendo $\cos. k\omega = \cos.\theta$.

§. 3. Pro ω igitur eiusmodi angulum aſſumi oportet, vt fiat $\cos. k\omega = \cos.\theta$, vnde quidem ſtatim deducitur $k\omega = \theta$, ideoque $\omega = \frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem coſinum habentes, qui praeter ipſum angulum θ ſunt $2\pi \pm \theta$, $4\pi \pm \theta$, $6\pi \pm \theta$ etc. atque adeo in genere $2i\pi \pm \theta$, denotante i omnes numeros integros, quaeſito noſtro ſatisfiet, faciendo $k\omega = 2i\pi \pm \theta$, vnde colligitur angulus $\omega = \frac{2i\pi \pm \theta}{k}$, ſicque pro ω nanciſceremus innumerabiles angulos ſatisfacientes, quorum autem ſufficiet

tot

tot assumsisse, quot exponens k continet vnitates; successiue igitur angulo ω sequentes tribuamus valores:

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \frac{8\pi+\theta}{k},\ \ldots\ \frac{2(k-1)\pi+\theta}{k}$$

Quod si ergo angulo ω successiue singulos istos valores, quorum numerus est $=k$, tribuamus, formula $1-2x\,\mathrm{cos.}\,\omega+xx$ omnes suppeditabit factores duplicatos nostri denominatoris $1-2x^k\,\mathrm{cos.}\,\theta+x^{2k}$, quorum numerus erit $=k$.

§. 4. Inuentis iam omnibus factoribus duplicatis nostri denominatoris, fractio $\frac{x^{m-1}}{1-2x^k\,\mathrm{cos.}\,\theta+x^{2k}}$ resolui debet in tot fractiones partiales, quarum denominatores sint ipsi isti factores duplicati, quorum numerus est k, ita vt in genere talis fractio partialis habitura sit talem formam:

$$\frac{A+Bx}{1-2x\,\mathrm{cos.}\,\omega+xx}$$

quam insuper resoluamus in binas simplices, etsi imaginarias, et cum sit

$$xx-2x\,\mathrm{cos.}\,\omega+1=(x-\mathrm{cos.}\,\omega+\sqrt{-1}\,\mathrm{sin.}\,\omega)(x-\mathrm{cos.}\,\omega-\sqrt{-1}\,\mathrm{sin.}\,\omega),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x-\mathrm{cos.}\,\omega-\sqrt{-1}\,\mathrm{sin.}\,\omega}+\frac{g}{x-\mathrm{cos.}\,\omega+\sqrt{-1}\,\mathrm{sin.}\,\omega},$$

ita vt totum resolutionis negotium huc redeat, vt ambo numeratores f et g determinentur; iis enim inuentis habebitur summa ambarum fractionum

$$=\frac{fx+gx-(f+g)\,\mathrm{cos.}\,\omega+\sqrt{-1}\,(f-g)\,\mathrm{sin.}\,\omega}{xx-2x\,\mathrm{cos.}\,\omega+1},$$

vbi igitur erit

$$B=f+g \quad\text{et}\quad A=(f-g)\sqrt{-1}\,\mathrm{sin.}\,\omega-(f+g)\,\mathrm{cos.}\,\omega.$$

§. 5. Per methodum igitur fractiones quascunque in fractiones simplices resoluendi statuamus

$$\frac{x^{m-1}}{1-2x^k \cos.\theta + x^{2k}} = \frac{f}{x-\cos.\omega - \sqrt{-1}\sin.\omega} + R,$$

vbi R complectatur omnes reliquas fractiones partiales. Hinc per $x-\cos.\omega-\sqrt{-1}\sin.\omega$ multiplicando habebitur

$$\frac{x^m - x^{m-1}(\cos.\omega + \sqrt{-1}\sin.\omega)}{1-2x^k\cos.\theta + x^{2k}} = fR(x-\cos.\omega - \sqrt{-1}\sin.\omega),$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x = \cos.\omega + \sqrt{-1}\sin.\omega$, vt membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra, quia formula $x-\cos.\omega-\sqrt{-1}\sin.\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita vt hinc nihil concludi posse videatur.

§. 6. Hic igitur vtamur regula notissima, et loco tam numeratoris quam denominatoris eorum differentialia scribamus, vnde nostra aequatio accipiet sequentem formam:

$$\frac{mx^{m-1}-(m-1)x^{m-2}(\cos.\omega+\sqrt{-1}\sin.\omega)}{-2kx^{k-1}\cos.\theta + 2kx^{2k-1}} =$$

$$\frac{mx^m-(m-1)x^{m-1}(\cos.\omega+\sqrt{-1}\sin.\omega)}{-2kx^k\cos.\theta + 2kx^{2k}} = f,$$

posito scilicet $x = \cos.\omega + \sqrt{-1}\sin.\omega$. Tum autem erit

$$x^m = \cos.m\omega + \sqrt{-1}\sin.m\omega \text{ et}$$
$$x^{m-1}(\cos.\omega+\sqrt{-1}\sin.\omega) = x^m = \cos.m\omega + \sqrt{-1}\sin.m\omega$$

et pro denominatore

$$x^k = \cos.k\omega + \sqrt{-1}\sin.k\omega \text{ et}$$
$$x^{2k} = \cos.2k\omega + \sqrt{-1}\sin.2k\omega;$$

vnde

vnde fit numerator

$$x^m = \cos. m\omega + \sqrt{-1} \sin. m\omega$$

et denominator

$$- 2k \cos.\theta \cos. k\omega + 2k \cos. 2k\omega$$
$$- 2k\sqrt{-1} \cos.\theta \sin. k\omega + 2k\sqrt{-1} \sin. 2k\omega.$$

§. 7. Pro denominatore reducendo recordemur, iam supra inuentum esse $\cos. k\omega = \cos.\theta$, vnde fit $\sin. k\omega = \sin.\theta$, tum vero

$$\cos. 2k\omega = \cos. 2\theta = 2\cos.\theta^2 - 1 \text{ et } \sin. 2k\omega = 2\sin.\theta\cos.\theta,$$

quibus valoribus adhibitis denominator noster erit

$$2k\cos.\theta^2 - 2k + 2k\sqrt{-1}\sin.\theta\cos.\theta = -2k\sin.\theta^2 + 2k\sqrt{-1}\sin.\theta\cos.\theta$$
$$= -2k\sin.\theta(\sin.\theta - \sqrt{-1}\cos.\theta),$$

quamobrem hoc valore adhibito habebimus

$$f = \frac{\cos. m\omega + \sqrt{-1}\sin. m\omega}{2k\sin.\theta(\sqrt{-1}\cos.\theta - \sin.\theta)}.$$

Simul vero hinc sine nouo calculo deducemus valorem g, quippe qui ab f ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g = \frac{\cos. m\omega - \sqrt{-1}\sin. m\omega}{-2k\sin.\theta(\sin.\theta + \sqrt{-1}\cos.\theta)}.$$

§. 8. Inuentis autem his litteris f et g pro litteris A et C colligemus primo

$$f + g = \frac{\cos.\theta\sin. m\omega - \sin.\theta\cos. m\omega}{k\sin.\theta} = \frac{\sin.(m\omega - \theta)}{k\sin.\theta},$$

tum vero erit

$$f - g = -\frac{\sqrt{-1}\cos.(m\omega - \theta)}{k\sin.\theta}.$$

Ex his igitur reperiemus

B =

$$B = \frac{\sin.(m\omega - \theta)}{k \sin.\theta} \text{ et}$$

$$A = \frac{\sin.\omega \cos.(m\omega - \theta) - \cos.\omega \sin.(m\omega - \theta)}{k \sin.\theta} = -\frac{\sin.((m\omega - \theta) - \omega)}{k \sin.\theta},$$

vbi ergo imaginaria ſponte ſe mutuo deſtruxerunt.

§. 9. Inuentis his valoribus A et B inueſtigari oportet integrale partiale $\int \frac{(A + Bx)\,dx}{1 - 2x \cos.\omega + xx}$, vbi, cum denominatoris differentiale ſit

$$2x\,dx - 2\,dx \cos.\omega = 2\,dx\,(x - \cos.\omega),$$

ſtatuamus

$$A + Bx = B(x - \cos.\omega) + C, \text{ eritque } C = A + B\cos.\omega,$$

hinc igitur erit

$$C = \frac{\cos.\omega \sin.(m\omega - \theta) - \sin.((m\omega - \theta) - \omega)}{k \sin.\theta}.$$

Quia vero

$$-\sin.(m\omega - \theta - \omega) = -\sin.(m\omega - \theta)\cos.\omega + \cos.(m\omega - \theta)\sin.\omega, \text{ erit}$$

$$C = \frac{\sin.\omega \cos.(m\omega - \theta)}{k \sin\theta}.$$

Hac ergo forma adhibita formula integranda $\frac{(A + Bx)\,dx}{1 - 2x\cos.\omega + xx}$ diſcerpatur in has duas partes:

$$\frac{B(x - \cos.\omega)\,dx}{1 - 2x\cos.\omega + xx} + \frac{C\,dx}{1 - 2x\cos.\omega + xx}.$$

Hic igitur prioris partis integrale manifeſto eſt

$$B\, l \sqrt{(1 - 2x\cos.\omega + xx)},$$

alterius vero partis facile patet integrale per arcum circuli expreſſum iri, cuius tangens ſit $\frac{x \sin.\omega}{1 - x\cos.\omega}$. Ad hoc integrale inueniendum ponamus

$$\int \frac{C\,dx}{1 - 2x\cos.\omega + xx} = D.\ A\,\text{tang.}\ \frac{x \sin.\omega}{1 - x\cos.\omega}$$

et

et ſumtis differentialibus, quia $d.$ A tang. t aequale eſt $\frac{dt}{1+tt}$, habebimus

$$\frac{C\,dx}{1-2x\cos.\omega+xx}=D.\frac{dx\sin.\omega}{1-2x\cos.\omega+xx}$$

vnde manifeſto fit

$$D=\frac{C}{\sin.\omega}=\frac{\cos.(m\omega-\theta)}{k\sin.\theta}.$$

§. 10. Subſtituamus igitur loco B et D valores modo inuentos et ex ſingulis factoribus denominatoris

$$1-2x^k\cos.\theta+x^{2k},$$

quorum forma eſt $1-2x\cos.\omega+xx$, oritur pars integralis conſtans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega-\theta)}{k\sin.\theta}l\sqrt{(1-2x\cos.\omega+xx)}+\frac{\cos.(m\omega-\theta)}{k\sin\theta}\text{A tang.}\frac{x\sin.\omega}{1-x\cos.\omega}$$

quae euaneſcit ſumto $x=0$. In hac igitur forma tantum opus eſt vt loco ω ſucceſſiue ſcribamus valores ſupra indicatos, ſcilicet:

$$\omega=\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi-\theta}{k},\ \frac{6\pi+\theta}{k},\ \text{etc.}$$

donec perueniatur ad $\frac{2(k-1)\pi+\theta}{k}$; tum enim ſumma omnium harum formarum praebebit totum integrale indefinitum formulae propoſitae.

§. 11. Poſtquam igitur integrale indefinitum elicuimus, nihil aliud ſupereſt, niſi vt in eo faciamus $x=\infty$, quo facto pars logarithmica, ob

$$\sqrt{(1-2x\cos.\omega+xx)}=x-\cos.\omega,$$

erit B $l(x-\cos.\omega)$. Eſt vero

$$l(x-\cos.\omega)=lx-\frac{\cos.\omega}{x}=lx,\text{ ob }\frac{\cos.\omega}{k}=0;$$

 quam-

quamobrem facto $x = \infty$ quaelibet pars logarithmica habebit hanc formam: $\frac{\sin.(m\omega - \theta)}{k \sin.\theta} l x$. Deinde pro partibus a circulo pendentibus, facto $x = \infty$ fit

$$\frac{x \sin.\omega}{1 - x \cos.\omega} = -\text{tang.}\,\omega = \text{tang.}\,(\pi - \omega),$$

sicque arcus, cuius haec est tangens, erit $= \pi - \omega$, hincque pars circularis quaecunque fiet $\frac{\cos.(m\omega - \theta)}{k \sin.\theta}(\pi - \omega)$.

§. 12. Cum quilibet valor anguli ω in genere hanc habeat formam: $\frac{2i\pi + \theta}{k}$, erit angulus

$$m\omega - \theta = \frac{2im\pi - \theta(k-m)}{k} \text{ et } \pi - \omega = \frac{\pi(k-2i) - \theta}{k}.$$

Ponamus breuitatis gratia

$$\frac{\theta(k-m)}{k} = \zeta \text{ et } \frac{m\pi}{k} = \alpha, \quad \text{vt sit } m\omega - \theta = 2i\alpha - \zeta,$$

vbi loco i scribi debent successiue numeri 0, 1, 2, 3, etc. vsque ad $k-1$. Hinc igtiur si omnes partes logarithmicas in vnam summam colligamus, ea ita repraesentari poterit:

$$\frac{l x}{k \sin.\theta}(-\sin.\zeta + \sin.(2\alpha - \zeta) + \sin.(4\alpha - \zeta) + \sin.(6\alpha - \zeta) \\ + \sin.(8\alpha - \zeta) - - - - + \sin.(2(k-1)\alpha - \zeta));$$

vbi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet, totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

§. 13. Ad hoc ostendendum ponamus

$$S = -\sin.\zeta + \sin.(2\alpha - \zeta) + \sin.(4\alpha - \zeta) + - - - - + \sin.(2(k-1)\alpha - \zeta)$$

multiplicemus vtrinque per $2 \sin.\alpha$, et cum sit

$$2 \sin.\alpha \sin.\phi = \cos.(\alpha - \phi) - \cos.(\alpha + \phi)$$

huius reductionis ope obtinebimus sequentem expressionem:

$$2\,S\,\text{fin.}\,\alpha = \text{cof.}(\alpha+\zeta) - \text{cof.}(\alpha-\zeta) + \text{cof.}(3\alpha-\zeta)$$
$$+\text{cof.}(\alpha-\zeta) \qquad -\text{cof.}(3\alpha-\zeta)$$
$$+\text{cof.}(5\alpha-\zeta)\,\cdots\cdots - \text{cof.}((2k-1)\alpha-\zeta)$$
$$-\text{cof.}(5\alpha-\zeta)\,\cdots\cdots$$

vnde deletis terminis se mutuo destruentibus habebitur

$$2\,S\,\text{fin.}\,\alpha = \text{cof.}(\alpha+\zeta) - \text{cof.}(2k-1)\alpha-\zeta).$$

§. 14. Ponamus hos duos angulos, qui sunt relicti, $\alpha+\zeta = p$ et $(2k-1)\alpha-\zeta = q$, eritque eorum summa $p+q = 2\alpha k$. Quia porro est $\alpha = \frac{m\pi}{k}$, erit $p+q = 2m\pi$, hoc est multiplo totius circuli peripheriae, ob m numerum integrum. Quare cum sit $q = 2m\pi - p$, erit cof. $q =$ cof. p; vnde patet summam inuentam nihilo esse aequalem, sicque manifestum est, omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x = \infty$ se mutuo destruere.

§. 15. Progrediamur igitur ad partes circulares, quarum forma generalis, vt vidimus, est $\frac{\text{cof.}(m\omega-\theta)}{k\,\text{fin.}\,\theta}(\pi-\omega)$, quae posito $\alpha = \frac{m\pi}{k}$ et $\zeta = \frac{\theta(k-m)}{k}$ fit

$$\frac{\text{cof.}(2i\alpha-\zeta)}{k\,\text{fin.}\,\theta}\left(\pi - \frac{2i\pi-\theta}{k}\right) = \frac{\text{cof.}(2i\alpha-\zeta)}{k\,\text{fin.}\,\theta}\left(\pi - \frac{2i\pi}{k} - \frac{\theta}{k}\right).$$

Hic ponatur porro $\frac{\pi}{k} = \beta$ et $\pi - \frac{\theta}{k} = \gamma$, vt forma generalis sit $\frac{\text{cof.}(2i\alpha-\zeta)}{k\,\text{fin.}\,\theta}(\gamma - 2i\beta)$. Quare si loco i scribamus ordine valores 0, 1, 2, 3, 4, vsque ad $k-1$, omnes partes circulares hanc progressionem constituent:

$$\frac{1}{k\,\text{fin.}\,\theta}(\gamma\,\text{cof.}\,\zeta + (\gamma-2\beta)\,\text{cof.}(2\alpha-\zeta) + (\gamma-4\beta)\,\text{cof.}(4\alpha-\zeta)$$
$$\cdots\cdots\ (\gamma-2(k-1)\beta)\,\text{cof.}(2(k-1)\alpha-\zeta)).$$

Ponamus

Ponamus igitur

$$S=\gamma\,\mathrm{cof.}\zeta+(\gamma-2\beta)\,\mathrm{cof.}(2\alpha-\zeta)+(\gamma-4\beta)\,\mathrm{cof.}(4\alpha-\zeta)$$
$$----(\gamma-2(k-1)\beta)\,\mathrm{cof.}(2(k-1)\alpha-\zeta).$$

vt fumma omnium partium circularium fit $\frac{S}{k\,\mathrm{fin.}\theta}$, quae ergo praebebit valorem quaefitum formulae integralis propofitae, cafu quo poft integrationem ftatuitur $x=\infty$, ita vt totum negotium in inueftigando valore ipfius S verfetur.

§. 16. Hunc in finem multiplicemus vtrinque per $2\,\mathrm{fin.}\,\alpha$, et cum in genere fit

$$2\,\mathrm{fin.}\,\alpha\,\mathrm{cof.}\,\phi=\mathrm{fin.}(\alpha+\phi)-\mathrm{fin.}(\phi-\alpha)$$

hac reductione in fingulis terminis facta perueniemus ad hanc aequationem:

$$2S\,\mathrm{fin.}\,\alpha=\gamma\,\mathrm{fin.}(\alpha+\zeta)+\gamma\,\mathrm{fin.}(\alpha-\zeta)+(\gamma-2\beta)\,\mathrm{fin.}(3\alpha-\zeta)$$
$$-(\gamma-2\beta)\,\mathrm{fin.}(\alpha-\zeta)-(\gamma-4\beta)\,\mathrm{fin.}(3\alpha-\zeta)$$
$$+(\gamma-4\beta)\,\mathrm{fin.}(5\alpha-\zeta)----+(\gamma-2(k-1)\beta)\,\mathrm{fin.}(2k-1)\alpha-\zeta)$$
$$-(\gamma-6\beta)\,\mathrm{fin.}(7\alpha-\zeta)-----$$

vbi praeter primum et vltimum terminum omnes reliqui contrahi poffunt, ita vt prodeat

$$2S\,\mathrm{fin.}\,\alpha=\gamma\,\mathrm{fin.}(\alpha+\zeta)+2\beta\,\mathrm{fin}(\alpha-\zeta)+2\beta\,\mathrm{fin.}(3\alpha-\zeta)+2\beta\,\mathrm{fin.}(5\alpha-\zeta)$$
$$--------.\ \mathrm{fin.}((2k-1)\alpha-\zeta)+$$

§. 17. Iam pro hac ferie fummanda ponamus porro

$$T=2\,\mathrm{fin.}(\alpha-\zeta)+2\,\mathrm{fin.}(3\alpha-\zeta)+2\,\mathrm{fin.}(5\alpha-\zeta)+---+2\,\mathrm{fin.}(2k-3)\alpha-\zeta)$$

vt habeamus

2 S

$2S \sin.\alpha = \gamma \sin.(\alpha + \zeta) + (\gamma - 2(k-1)\beta)\sin.((2k-1)\alpha - \zeta) + \beta T.$

Iam multiplicemus, vt hactenus, per sin. α, et cum sit

$$2 \sin.\alpha \sin.\phi = \cos.(\phi - \alpha) - \cos.(\phi + \alpha),$$

facta hac reductione nanciscimur

$$T \sin.\alpha = +\cos.\zeta + \cos.(2\alpha - \zeta) + \cos.(4\alpha - \zeta) + \text{---} + \cos.(2(k-2)\alpha - \zeta)$$
$$-\cos.(2\alpha - \zeta) - \cos.(4\alpha - \zeta) - \cos.(6\alpha - \zeta) - \text{----} - \cos.(2(k-1)\alpha - \zeta)$$

vnde deletis terminis, quae se mutuo destruunt, remanebit tantum ista expressio:

$$T \sin.\alpha = \cos.\zeta - \cos.(2(k-1)\alpha - \zeta).$$

Cum igitur sit $\alpha = \frac{m\pi}{k}$ erit

$$2(k-1)\alpha = 2m\pi - \frac{2m\pi}{k},$$

cuius loco scribere licet $-\frac{2m\pi}{k}$, vnde ob $\zeta = \frac{\theta(k-m)}{k}$ erit

$$T \sin.\alpha = \cos.\frac{\theta(k-m)}{k} - \cos.\left(\frac{2m\pi + \theta(k-m)}{k}\right).$$

§. 18. Nunc vero notetur in genere esse

$$\cos.p - \cos.q = 2 \sin.\frac{q+p}{2} \sin.\frac{q-p}{2},$$

quare cum sit

$$p = \frac{\theta(k-m)}{k} \text{ et } q = \frac{2m\pi + \theta(k-m)}{k}, \text{ erit}$$

$$\frac{q+p}{2} = \frac{m\pi + \theta(k-m)}{k} \text{ et } \frac{q-p}{2} = \frac{m\pi}{k},$$

vnde sequitur fore

$$T \sin.\alpha = 2 \sin.\left(\frac{m\pi + \theta(k-m)}{k}\right) \sin.\frac{m\pi}{k}, \text{ ideoque}$$

$$T = 2 \sin.\left(\frac{m\pi + \theta(k-m)}{k}\right), \text{ ob } \alpha = \frac{m\pi}{k}.$$

§. 19. Hoc igitur valore T inuento reperiemus porro

$$2S \sin.\alpha = \gamma \sin.(\alpha + \zeta) + (\gamma - 2(k-1)\beta)\sin.((2k-1)\alpha - \zeta)$$
$$+ 2\beta \sin.\left(\frac{m\pi + \theta(k-m)}{k}\right)$$

 quae

quae ob $\frac{m\pi+\theta(k-m)}{k} = \alpha + \zeta$ reducitur ad hanc formam:

$$2\,S\,\text{sin.}\,\alpha = (\gamma+2\beta)\,\text{sin.}\,(\alpha+\zeta) + (\gamma-2(k-1)\beta)\,\text{sin.}\,((2k-1)\alpha-\zeta),$$

quae ita repraesentari potest:

$$2\,S\,\text{sin.}\,\alpha = (\gamma+2\beta)(\text{sin.}\,(\alpha+\zeta) + \text{sin.}\,((2k-1)\alpha-\zeta)) - 2k\beta\,\text{sin.}\,((2k-1)\alpha-\zeta),$$

vbi pro parte priore, ob

$$\text{sin.}\,p + \text{sin.}\,q = 2\,\text{sin.}\,\frac{p+q}{2}\,\text{cos.}\,\frac{p-q}{2}, \text{ erit}$$

$$\frac{p+q}{2} = \alpha k \text{ et } \frac{p-q}{2} = (k-1)\alpha - \zeta;$$

vnde pars ipsa prior fit

$$2(\gamma+2\beta)\,\text{sin.}\,\alpha k\,\text{cos.}\,((k-1)\alpha-\zeta),$$

vbi cum sit $\alpha k = m\pi$, erit $\text{sin.}\,\alpha k = 0$, ita vt tantum supersit

$$2\,S\,\text{sin.}\,\alpha = -2\beta k\,\text{sin.}\,((2k-1)\alpha-\zeta) \text{ hincque}$$

$$S = -\frac{\beta k\,\text{sin.}\,((2k-1)\alpha-\zeta)}{\text{sin.}\,\alpha}. \text{ Est vero}$$

$$(2k-1)\alpha - \zeta = 2m\pi - \frac{m\pi}{k} - \frac{\theta(k-m)}{k};$$

omisso termino $2m\pi$ erit igitur

$$S = +\frac{\pi\,\text{sin.}\,\frac{(m\pi+\theta(k-m))}{k}}{\text{sin.}\,\frac{m\pi}{k}}$$

ideoque valor quaesitus erit

$$\frac{S}{k\,\text{sin.}\,\theta} = +\frac{\pi\,\text{sin.}\,\frac{(m\pi+\theta(k-m))}{k}}{k\,\text{sin.}\,\theta\,\text{sin.}\,\frac{m\pi}{k}}$$

quae forma reducitur ad hanc:

$$\frac{\pi\,\text{sin.}\,\frac{(m(\pi-\theta)+k\theta)}{k}}{k\,\text{sin.}\,\theta\,\text{sin.}\,\frac{m\pi}{k}}.$$

§. 20.

§. 20 Contemplemur hic ante omnia casum quo $\theta = \frac{\pi}{2}$, et formula integralis proposita abit in hanc:

$$\int \frac{x^{m-1}\,dx}{1 + x^{2k}}$$

cuius ergo valor, si post integrationem ponatur $x = \infty$, euadet

$$= \frac{\pi\ \text{sin.}\ (\frac{\pi}{2} + \frac{m\pi}{2k})}{k\ \text{sin.}\ \frac{m\pi}{k}} = \frac{\pi\ \text{cos.}\ \frac{m\pi}{2k}}{k\ \text{sin.}\ \frac{m\pi}{k}}.$$

Quia igitur est

$$\text{sin.}\ \frac{m\pi}{k} = 2\ \text{sin.}\ \frac{m\pi}{2k}\ \text{cos.}\ \frac{m\pi}{2k},$$

prodibit iste valor $= \frac{\pi}{2\,k\ \text{sin.}\ \frac{m\pi}{2k}}$, qui valor egregie conuenit cum eo, quem non ita pridem pro formula $\int \frac{x^{m-1}\,dx}{1 + x^{k}}$ assignauimus, si quidem loco k scribatur $2k$.

§. 21. Euoluamus etiam casum quo $\theta = \pi$, et formula nostra integralis $\int \frac{x^{m-1}\,dx}{(1 + x^{k})^{2}}$, cuius ergo, facto $x = \infty$, valor erit

$$\frac{\pi\ \text{sin.}\ (\frac{m(\pi - \theta)}{k} + \theta)}{k\ \text{sin.}\ \theta\ \text{sin.}\ \frac{m\pi}{k}} = \frac{\pi}{k\ \text{sin.}\ \frac{m\pi}{k}} \cdot \frac{\text{sin.}\ \frac{m(\pi - \theta)}{k} + \theta}{\text{sin.}\ \theta}.$$

Huius autem posterioris fractionis, casu $\theta = \pi$, tam numerator quam denominator euanescit; quare, vt eius verus valor eruatur, loco vtriusque eius differentiale scribamus, quo facto ista fractio abibit in hanc: $\frac{d\theta(1 - \frac{m}{k})\ \text{cos.}\ (\frac{m(\pi - \theta)}{k} + \theta)}{d\theta\ \text{cos.}\ \theta}$, cuius valor facto $\theta = \pi$ nunc manifesto est $1 - \frac{m}{k}$; sicque valor in-

integralis quaesitus erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k\,\text{sin.}\,\frac{m\pi}{k}}$, prorsus vti in superiore dissertatione inuenimus.

§. 22. Quo autem valorem generalem inuentum commodiorem reddamus, ponamus $\pi-\theta=\eta$, fietque $\text{sin.}\,\theta=\text{sin.}\,\eta$ et $\text{cos.}\,\theta=-\text{cos.}\,\eta$; tum vero erit angulus

$$\frac{m(\pi-\theta)}{k}+\theta=\frac{m\pi}{k}+\pi-\eta,$$

cuius sinus est $\text{sin.}\left(1-\frac{m}{k}\right)\eta$, vnde valor quaesitus nostrae formulae erit $\frac{\pi\,\text{sin.}\left(1-\frac{m}{k}\right)\eta}{k\,\text{sin.}\,\eta\,\text{sin.}\,\frac{m\pi}{k}}$, atque hinc tandem sequens adepti sumus

Theorema.

§. 23. *Si haec formula integralis:*

$$\int\frac{x^{m-1}dx}{1+2x^k\,\text{cos.}\,\eta+x^{2k}}$$

a termino $x=0$ *vsque ad terminum* $x=\infty$ *extendatur, eius valor erit* $=\frac{\pi\,\text{sin.}\left(1-\frac{m}{k}\right)\eta}{k\,\text{sin.}\,\eta\,\text{sin.}\,\frac{m\pi}{k}}$, *siue cum sit*

$$\text{sin.}\left(1-\frac{m}{k}\right)\eta=\text{sin.}\,\eta\,\text{cos.}\,\frac{m\eta}{k}-\text{cos.}\,\eta\,\text{sin.}\,\frac{m\eta}{k},$$

iste valor etiam hoc modo exprimi potest:

$$\frac{\pi\,\text{cos.}\,\frac{m\eta}{k}}{k\,\text{sin.}\,\frac{m\pi}{k}}-\frac{\pi\,\text{sin.}\,\frac{m\eta}{k}}{k\,\text{tang.}\,\eta\,\text{sin.}\,\frac{m\pi}{k}}.$$

§. 24

§. 24. Consideremus nunc alio modo hanc formulam integralem: $\int \frac{x^{m-1}\,dx}{1+2x^k \operatorname{cos.} \eta + x^{2k}}$, cuius valor a termino $x=0$ vsque ad $x=1$ ponatur $=P$, eiusdem vero valor ab $x=1$ vsque ad $x=\infty$ ponatur $=Q$, ita vt $P+Q$ exhibere debeat ipsum valorem ante inuentum. Nunc vero pro valore Q inueniendo ponamus $x=\frac{1}{y}$ et formula nostra ita repraesentata:

$$\frac{x^m}{1+2x^k \operatorname{cos.} \eta + x^{2k}} \frac{dx}{x},$$

ob $\frac{dx}{x} = -\frac{dy}{y}$ induet hanc formam:

$$-\int \frac{y^{-m}}{1+2y^{-k} \operatorname{cos.} \eta + y^{-2k}} \frac{dy}{y} = -\int \frac{y^{2k-m-1}\,dy}{y^{2k}+2y^k \operatorname{cos.} \eta + 1}$$

cuius valor a termino $y=1$ vsque ad $y=0$ extendi debet. Commutatis igitur his terminis habebimus

$$Q = +\int \frac{y^{2k-m-1}\,dy}{y^{2k}+2y^k \operatorname{cos.} \eta + 1}$$

a termino $y=0$ vsque ad $y=1$.

§. 25. Quia in vtraque forma pro P et Q eadem conditio integrationis praescribitur, a termino 0 vsque ad 1, nihil impedit quo minus in posteriore loco y scribamus x, vnde pro $P+Q$ habebimus hanc formam integralem:

$$\int \frac{x^{m-1}+x^{2k-m-1}}{1+2x^k \operatorname{cos.} \eta + x^{2k}}\,dx,$$

cuius valor, a termino $x=0$ vsque ad $x=1$ extensus, aequabitur huic expressioni: $\frac{\pi \operatorname{fin.} (1-\frac{m}{k})\eta}{k \operatorname{fin.} \eta \operatorname{fin.} \frac{m\pi}{k}}$. Comparatis igitur

 his

his binis formulis integralibus nanciscemur sequens Theorema notatu maxime dignum.

Theorema.

§. 26. *Haec formula integralis:*

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \operatorname{cos.} \eta + x^{2k}} dx,$$

a termino $x = 0$ *vsque ad terminum* $x = 1$ *extensa, aequalis est huic formulae integrali:* $\int \frac{x^{m-1}\, dx}{1 + 2x^k \operatorname{cos.} \eta + x^{2k}}$, *a termino* $x = 0$ *vsque ad terminum* $x = \infty$ *extensae: vtriusque enim valor erit* $\frac{\pi \operatorname{sin.} (1 - \frac{m}{k}) \eta}{k \operatorname{sin.} \eta \operatorname{sin.} \frac{m\pi}{k}}$.

§. 27. Quod si hanc fractionem: $\frac{\operatorname{sin.} \eta}{1 + 2x^k \operatorname{cos.} \eta + x^{2k}}$ in seriem infinitam euoluamus, quae sit,

$$\operatorname{sin.} \eta + A x^k + B x^{2k} + C x^{3k} + D x^{4k} + E x^{5k} \text{ etc.}$$

per denominatorem multiplicando perueniemus ad hanc expressionem infinitam:

$$\begin{aligned}\operatorname{sin.} \eta = \operatorname{sin.} \eta + {} & A\, x^k && + B\, x^{2k} && + C\, x^{3k} && + D\, x^{4k} && + E\, x^{5k} && + F\, x^{6k} + \text{etc.}\\ & + 2 \operatorname{sin.} \eta \operatorname{cos.} \eta && + 2A \operatorname{cos.} \eta && + 2B \operatorname{cos.} \eta && + 2C \operatorname{cos.} \eta && + 2D \operatorname{cos.} \eta && + 2E \operatorname{cos.} \eta + \text{etc.}\\ & && + \operatorname{sin.} \eta && + A && + B && + C && + D \text{ etc.}\end{aligned}$$

vnde singulis terminis ad nihilum reductis reperiemus

1° $A + 2 \operatorname{sin.} \eta \operatorname{cos.} \eta = 0$, hincque $A = - \operatorname{sin.} 2\eta$
2° $B + 2A \operatorname{cos.} \eta + \operatorname{sin.} \eta = 0$, vnde fit $B = \operatorname{sin.} 3\eta$
3° $C + 2B \operatorname{cos.} \eta + A = 0$, vnde fit $C = - \operatorname{sin.} 4\eta$
4° $D + 2C \operatorname{cos.} \eta + B = 0$, vnde fit $D = \operatorname{sin.} 5\eta$
etc. etc.

ita

ita vt nostra fractio $\frac{\sin.\eta}{1+2x^k\cos.\eta+x^{2k}}$ resoluatur in hanc seriem:

$$\sin.\eta - x^k\sin.2\eta + x^{2k}\sin.3\eta - x^{3k}\sin.4\eta + x^{4k}\sin.5\eta \text{ etc.}$$

§. 28. Multiplicemus nunc hanc seriem per $x^{m-1}dx + x^{2k-m-1}dx$ et post integrationem faciamus $x=1$, vt obtineamus valorem huius formulae:

$$\sin.\eta\int\frac{x^{m-1}+x^{2k-m-1}}{1+2x^k\cos.\eta+x^{2k}}dx$$

pro casu $x=1$, hocque modo perueniemus ad geminas sequentes series:

$$\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{ etc.}$$

$$\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{ etc.}$$

Aggregatum igitur harum duarum serierum iunctim sumtarum aequabitur huic valori: $\frac{\pi\sin.(1-\frac{m}{k})\eta}{k\sin.\frac{m\pi}{k}}$, vnde subiungamus adhuc istud Theorema:

Theorema.

§. 29. *Si η denotet angulum quemcunque, litterae vero* m *et* k *pro lubitu accipiantur, ex iisque binae sequentes series formentur:*

$$P=\frac{\sin.\eta}{m}-\frac{\sin.2\eta}{m+k}+\frac{\sin.3\eta}{m+2k}-\frac{\sin.4\eta}{m+3k}+\frac{\sin.5\eta}{m+4k}-\text{ etc.}$$

$$Q=\frac{\sin.\eta}{2k-m}-\frac{\sin.2\eta}{3k-m}+\frac{\sin.3\eta}{4k-m}-\frac{\sin.4\eta}{5k-m}+\frac{\sin.5\eta}{6k-m}-\text{ etc.}$$

neutrius quidem summa exhiberi potest, vtriusque autem iunctim sumtae summa erit

P +

$$P+Q=\frac{\pi \sin.\left(1-\frac{m}{k}\right)\eta}{k \sin.\frac{m\pi}{k}}$$

Corollarium.

§. 30. Quod ſi ergo angulum η infinite paruum capiamus, vt fiat

$$\sin.\eta=\eta;\ \sin.2\eta=2\eta;\ \sin.3\eta=3\eta;\ \text{etc.}$$

quia in formula ſummae fiet

$$\sin.\left(1-\frac{m}{k}\right)\eta=\left(1-\frac{m}{k}\right)\eta;$$

ſi vtrinque per η diuidamus, obtinebimus ſequentem ſeriem geminatam:

$$\frac{1}{m}-\frac{2}{m+k}+\frac{3}{m+2k}-\frac{4}{m+3k}+\frac{5}{m+4k}-\text{etc.}$$

$$\frac{1}{2k-m}-\frac{2}{3k-m}+\frac{3}{4k-m}-\frac{4}{5k-m}+\frac{5}{6k-m}-\text{etc.}$$

cuius ergo ſumma erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k\sin.\frac{m\pi}{k}}$; vbi notetur, ambas iſtas ſeries non incongrue in hanc ſimplicem contrahi poſſe:

$$\frac{2k}{m(2k-m)}-\frac{8k}{(m+k)(3k-m)}+\frac{18k}{(m+2k)(4k-m)}-\frac{32k}{(m+3k)(5k-m)}+\text{etc.}$$

vbi numeratores ſunt numeri quadrati duplicati.

§. 31. Formulae autem, quarum valores hactenus inuenimus, multo concinnius et elegantius exprimi poſſunt, ſi loco exponentis m ſcribamus $k-n$, tum enim in valore integrali inuento fiet $\left(1-\frac{m}{k}\right)\eta=\frac{n\eta}{k}$; at vero pro denominatore fiet $\frac{m\pi}{k}=\pi-\frac{n\pi}{k}$, cuius ſinus erit $\sin.\frac{n\pi}{k}$; ſicque noſtra formula inuenta hanc induet formam: $\frac{\pi\sin.\frac{n\eta}{k}}{k\sin.\eta\sin.\frac{n\pi}{k}}$, quae ergo exprimet valorem huius formulae integralis:

$\int x$

$$\int \frac{x^{k-n-1}\,dx}{1+2x^k \operatorname{cof.} \eta + x^{2k}},$$

ab $x=0$ vsque ad $x=\infty$, vt et huius formulae:

$$\int \frac{x^{k-n-1}+x^{k+n-1}\,dx}{1+x^k \operatorname{cof.} \eta + x^{2k}},$$

a termino $x=0$ vsque ad terminum $x=1$; et quia vtriusque valor eſt $\frac{\pi \operatorname{fin.} \frac{n\eta}{k}}{k \operatorname{fin.} \eta \operatorname{fin.} \frac{n\pi}{k}}$, perſpicuum eſt eum manere eundem, etſi loco n ſcribatur $-n$, ex quo prior formula ita repraeſentari poterit:

$$\int \frac{x^{k\pm n-1}}{1+x^k \operatorname{cof.} \eta + x^{2k}};$$

at poſterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

§. 32. Ponendo $m=k-n$ etiam ſeries noſtra geminata pulchriorem accipiet faciem; habebitur enim

$$\frac{\operatorname{fin.} \eta}{k-n} - \frac{\operatorname{fin.} 2\eta}{2k-n} + \frac{\operatorname{fin.} 3\eta}{3k-n} - \frac{\operatorname{fin.} 4\eta}{4k-n} + \text{etc.}$$
$$\frac{\operatorname{fin.} \eta}{k+n} - \frac{\operatorname{fin.} 2\eta}{2k+n} + \frac{\operatorname{fin.} 3\eta}{3k+n} - \frac{\operatorname{fin.} 4\eta}{4k+n} + \text{etc}$$

cuius ergo ſumma erit $\frac{\pi \operatorname{fin.} \frac{n\eta}{k}}{k \operatorname{fin.} \frac{n\pi}{k}}$. Tum vero ſi hae geminae ſeries in vnam contrahantur et vtrinque per $2k$ diuidatur, obtinebitur ſequens ſummatio memoratu digna:

$$\frac{\pi \operatorname{fin.} \frac{n\eta}{k}}{2kk \operatorname{fin.} \frac{n\pi}{k}} = \frac{\operatorname{fin.} \eta}{kk-nn} - \frac{2 \operatorname{fin.} 2\eta}{4kk-nn} + \frac{3 \operatorname{fin.} 3\eta}{9kk-nn} - \frac{4 \operatorname{fin.} 4\eta}{16kk-nn} + \text{etc.}$$

§. 33. Quod si haec postrema series differentietur, sumendo solum angulum η variabilem, ob

$d.\ \text{sin.}\frac{n\eta}{k} = \frac{n\,d\eta}{k}\ \text{cos.}\frac{n\eta}{k}$ habebimus

$$\frac{\pi n\ \text{cos.}\frac{n\eta}{k}}{2k^3\ \text{sin.}\frac{n\pi}{k}} = \frac{\text{cos.}\,\eta}{kk-nn} - \frac{4\,\text{cos.}\,2\eta}{4kk-nn} + \frac{9\,\text{cos.}\,3\eta}{9kk-nn} - \frac{16\,\text{cos.}\,4\eta}{16kk-nn} + \text{etc.}$$

Vnde si sumatur $\eta = 0$, orietur ista summatio:

$$\frac{\pi n}{2k^3\ \text{sin.}\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.}$$

sin autem sumatur $\eta = 90° = \frac{\pi}{2}$, erit $\text{cos.}\,\eta = 0$, $\text{cos.}\,2\eta = -1$, $\text{cos.}\,3\eta = 0$, $\text{cos.}\,4\eta = +1$ etc. vnde nascitur sequens series:

$$\frac{n\pi\ \text{cos.}\frac{n\pi}{2k}}{2k^3\ \text{sin.}\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

Quia autem $\text{sin.}\frac{n\pi}{k} = 2\,\text{sin.}\frac{n\pi}{2k}\,\text{cos.}\frac{n\pi}{2k}$, erit eiusdem seriei summa $\frac{n\pi}{4k^3\ \text{sin.}\frac{n\pi}{2k}}$.

§. 34. At si series illa §. 32 exhibita in $d\eta$ ducatur et integretur, ob

$\int d\eta\ \text{sin.}\frac{n\eta}{k} = -\frac{k}{n}\ \text{cos.}\frac{n\eta}{k}$, erit

$$C - \frac{\pi\ \text{cos.}\frac{n\eta}{k}}{2nk\ \text{sin.}\frac{n\pi}{k}} = -\frac{\text{cos.}\,\eta}{kk-nn} + \frac{\text{cos.}\,2\eta}{4kk-nn} - \frac{\text{cos.}\,3\eta}{9kk-nn} + \frac{\text{cos.}\,4\eta}{16kk-nn} + \text{etc.}$$

Vt autem hic constantem addendam C definiamus, sumamus $\eta = 0$, fietque

$$C - \frac{\pi}{2nk\ \text{sin.}\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.}$$

quare

quare si huius seriei summa aliunde pateat, constans C definiri poterit. Series autem haec in sequentem geminatam resolui potest:

$$2nC - \frac{\pi}{k\sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$

$$- \frac{1}{k-1} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} - \text{etc.}$$

§. 35. Cum igitur in *Introductione in Analysin Infinitorum* pag. 142. ad hanc peruenissem seriem:

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.}$$

$$= \frac{\pi}{2kn\sin.\frac{n\pi}{k}} - \frac{1}{2nn}$$

(hic scilicet loco litterarum ibi adhibitarum m et n scripsi n et k) hoc valore adhibito nostra aequatio erit

$$C - \frac{\pi}{2nk\sin.\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk\sin.\frac{n\pi}{k}},$$

vnde fit $C = \frac{1}{2nn}$. Hinc ergo habebimus istam summationem:

$$\frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\cos.\eta}{kk-nn} - \frac{\cos.2\eta}{4kk-nn}$$

$$+ \frac{\cos.3\eta}{9kk-nn} - \frac{\cos.4\eta}{16kk-nn} + \text{etc.}$$

quae series vtique omni attentione digna videtur.

THEOREMATA QVAEDAM ANALYTICA QVORVM DEMONSTRATIO ADHVC DESIDERATVR.

§. 1.

In Analyſi diophantea, quae circa proprietates numerorum verſatur, notiſſimum eſt, plurima occurrere theoremata, de quorum veritate dubitare non licet, etiamſi ea demonſtratione rigida confirmare non valeamus. In Geometria autem nemo adhuc eiusmodi theoremata in medium produxit, quorum vel veritatem vel falſitatem demonſtrare non liceat. At vero in Analyſi ſublimiori iam dudum etiam eiusmodi theoremata ſe mihi obtulerunt, quorum demonſtrationem nullo modo etiamnunc inuenire potui, etiamſi eorum veritas nequaquam in dubium vocari videatur. Talia igitur theoremata vtique ſummam attentionem merentur, cum nullum plane ſit dubium, quin ſi eorum demonſtrationem adhuc fruſtra anquiſitam detexeremus, inde maximi momenti incrementa in Analyſin ſint redundatura.

§. 2. Inter huiusmodi autem veritates analyticas merito primum locum tribuo inſigni illi proprietati quantitatum imaginariarum, quod, vbicunque tales quantitates natura ſua impoſſibiles occurrant, eae ſemper in formula hac

$a+b\sqrt{-1}$ comprehendi queant. Huic quidem veritati innititur resolutio omnium aequationum algebraicarum, quippe quarum radices nisi fuerint reales, omnes in tali formula $a+b\sqrt{-1}$ contineri perhibentur, id quod etiam illustris *d'Alembert* demonstratione perquam ingeniosa confirmauit, quae autem quoniam ex consideratione infinite paruorum est petita, haud immerito adhuc demonstratio planior ex ipsa natura imaginariorum petenda desideratur. Praeterea vero ista demonstratio tantum ad expressiones algebraicas patet, cum tamen aeque certum sit, eam etiam in omnis generis quantitatibus transcendentibus locum habere, vbi ratiocinium, quo vir celeberr. est vsus, non semper adhiberi potest, id quod operae pretium erit clarius ostendisse.

§. 3 Consideretur curua algebraica ex quotcunque ramis fuerit composita, cuiusmodi sit ramus FNLMH, qui ad axem AK relatus, postquam ab F dextrorsum vsque ad L processerit, hinc iterum sinistrorsum per LMH porrigatur; ita vt, si applicata KL hanc curuam in extremitate L tangat, abscissae cuilibet AP, minori quam AK, duplex respondeat applicata PM et PN. Vnde si ponatur abscissa $AP = x$, applicata y duplicem habebit valorem, ex tali aequatione quadratica: $yy = 2py - q$ determinandum, ita vt hinc sit altera applicata $PM = p - \sqrt{(pp-q)}$, altera vero $PN = p + \sqrt{(pp-q)}$, vbi pro indole curuae litterae p et q functiones quascunque abscissae x denotare possunt. Quamdiu igitur fuerit $pp > q$, reuera gemina orietur applicata PM et PN. Dum autem abscissa x vsque in K augetur, vbi fiat $pp = q$, ibi ambae applicatae in vnam KL coalescent, ita vt hic applicata KL euadat curuae tangens. Quod si ergo, abscissam x vlterius augendo, fiat $q > pp$, ambae applicatae euadent imaginariae. Tab. I. Fig. 3.

 Vnde

Vnde intelligitur, ſi capiatur abſciſſa $AX > AK$, in hoc loco nullam prorſus dari applicatam, ſeu rectam in hoc loco perpendicularem XY, vtrinque etiam in infinitum productam, nusquam curuae FLH eſſe occurſuram, id quod more loquendi in Analyſi recepto idem ſignificat ac applicatam in hoc loco X eſſe imaginariam; vnde ſimul notio imaginariorum, vti in Analyſi adpellantur, clarius intelligitur. Cum enim haec applicata XY curuae nusquam occurrat, etiamſi a puncto X, vbi eſt $= 0$, tam ſurſum vsque in infinitum poſitiuum, quam deorſum vsque in infinitum negatiuum continuetur: euidens eſt eius valorem inuentum neque eſſe $= 0$, neque maiorem quam 0, neque minorem quam 0, qua conditione definitio ipſa quantitatum imaginariarum continetur. Quod ſi ergo pro hoc loco ſumamus fieri $q = pp + rr$, gemina expreſſio applicatae euadet $y = p \pm r \sqrt{-1}$.

§. 4. Hic igitur quaeritur, num hinc certo in genere concludi poſſit, quotiescunque imaginaria occurrant, ea ſemper huiusmodi formula: $p \pm r \sqrt{-1}$ exprimi poſſe. Primo enim haec demonſtratio tantum ex ramo FLH eſt petita, dum tota curua aequatione inter x et y contenta fortaſſe plures inſuper alios ramos inuoluat, quos in hoc negotio penitus negligere fortaſſe non liceat. Hanc autem obiectionem Vir excell. vtique ipſe praeuidit, dum hoc ratiocinium tantum ad portiunculam curuae infinite paruam NLM extendit, vbi vlteriorem ramorum extenſionem tuto negligere liceat, quod autem non adeo in aprico ſitum videtur, vt non planiorem demonſtrationem a tali conceptu immunem merito deſiderare queamus. Tum vero etiam hinc plus non ſequeretur, quam applicatas XY, extremae KL infinite propinquas, tali formula $p \pm r \sqrt{-1}$ exprimi poſſe;

ac

ac non immerito dubitare liceret, an pro interuallis maioribus KX etiam applicatae tali formula comprehendi queant, et annon reliquae curuae partes hactenus neglectae indolem imaginarii in his locis penitus immutare valeant.

§. 5. Praeterea vero ista consideratio tantum ad aequationes et curuas algebraicas est accommodata, in quibus vtique alii rami non dantur, nisi qui vel in se redeant, vel vtrinque in infinitum excurrant, ita vt circa terminum L portio curuae hic semper binas portiones LM et LN exhibeat, vnde aequatio illa quadratica $yy = 2py - q$ est nata, cui tota demonstratio innititur. At vero inter curuas transcendentes eiusmodi rami occurrunt, qui neque vtrinque in infinitum protenduntur, neque in se redeunt, sed subito in quopiam puncto terminantur. Talem casum praebet curua transcendens hac aequatione contenta: $y = a + \frac{bx}{l(c-x)}$, ex qua sequitur, singulis abscissis vnicam tantum applicatam respondere. Posito enim $x = 0$, fit $y = a$; ac si abscissa x continuo augeatur vsque ad valorem $x = c$, perpetuo vnica dabitur applicata; sumta vero abscissa $x = c$, ob $l(c-x) = -\infty$, fiet applicata in hoc loco $y = a$. Statim autem atque abscissa x vltra c augetur, applicata subito fiet imaginaria, propterea quod logarithmi quantitatum negatiuarum certo sunt imaginarii; quare sumta abscissa $x > c$, applicata y, etiamsi vtrinque in infinitum producatur, curuae tamen nostrae nusquam occurret. Hoc autem casu ratio supra allegata et naturae aequationis quadraticae innixa, penitus cessat, ita vt hic merito dubitare possimus, an ista applicata imaginaria etiam in formula $p + q\sqrt{-1}$ comprehendi queat. Saltem hic agnoscere debemus, istud theorema alia demonstratione indigere, ideoque maxime optandum esse vt talis

aequa-

aequatio immediate ex ipsa natura imaginariorum derivetur.

§. 6. Ante autem quam hoc argumentum deferam, ostendisse iuuabit, quomodo omnia plane imaginaria singulari prorsus ratione per circulum repraesentari possint. Ex

Tab. I. Fig. 4. puncto A, pro principio axis AB assumto, erigatur perpendiculum $AC = a$; centro C radio $CM = c$ describatur circulus, ac posita abscissa quacunque $AP = x$ eique respondente applicata $PM = y$ erit

$$y = AC + QM = AC + \sqrt{(CM^2 - CQ^2)} = a + \sqrt{(cc - xx)},$$

ita vt eius valor semper sit realis quamdiu abscissa x minor capitur quam radius c, simulac vero abscissa x radium c superat, veluti si sumatur $x = AX$, tum applicata XY certe erit imaginaria. At vero, quanquam ob hanc ipsam caussam applicata exhiberi nequit, tamen determinatum habet valorem imaginarium (iam enim euictum est, notionem determinati notioni imaginarii non aduersari). Quoniam enim ponitur $x > c$, statuatur $xx = cc + bb$, vt fiat $\sqrt{cc - xx} = b\sqrt{-1}$, ideoque applicata ista imaginaria $XY = a + b\sqrt{-1}$. Quare cum formula $a + b\sqrt{-1}$ omnes plane quantitates imaginarias contineat, eas per huiusmodi applicatam determinatam XY ad circulum quendam pertinentem repraesentare licebit. Posito scilicet perpendiculo $AC = a$, centro C, radio pro arbitrio assumto c, describatur circulus, ac sumatur abscissa $AX = \sqrt{(bb + cc)}$, tum enim applicata imaginaria XY istam formulam $a + b\sqrt{-1}$; exhibebit sicque mirabili quodam modo omnes adeo formulas imaginarias quasi geometrice construere licebit.

§. 7.

§. 7. Operae pretium erit hoc exemplo quodam declaraſſe. Quaeramus ſcilicet arcum circuli, cuius ſinus duplo maior ſit ſinu toto, qui ergo certe erit imaginarius. Poſito ergo ſinu toto $= 1$ integrari debet formula $\frac{dx}{\sqrt{(1-xx)}}$, ita vt integrale euaneſcat poſito $x = 0$, tum vero ſumi debebit $x = 2$, et valor integralis dabit ipſum arcum. Hunc in finem formulae differentiali $\frac{dx}{\sqrt{(1-xx)}}$ tribuamus hanc formam: $\frac{dx\sqrt{-1}}{\sqrt{(xx-1)}}$; conſtat autem eſſe $\int\frac{dx}{\sqrt{(xx-1)}} = \frac{l(x+\sqrt{(xx-1)})}{\sqrt{-1}}$, vnde poſito $x = 2$ arcus quaeſitus erit

$$= \sqrt{-1}\, l\frac{2+\sqrt{3}}{\sqrt{-1}} = \sqrt{-1}\, l(2+\sqrt{3}) - \sqrt{-1}\, l\sqrt{-1}.$$

Nouimus autem huius poſtremi membri valorem eſſe $\frac{\pi}{2}$, vnde arcus circuli, cuius ſinus $= 2$, erit $\frac{\pi}{2} + \sqrt{-1}\, l(2+\sqrt{3})$. Quamobrem vt huic arcui imaginario aequalem applicatam XY exhibeamus, in noſtra figura capiatur interuallum $AC = \frac{\pi}{4}$, ac deſcripto circulo radii $CM = c = 1$, quia c arbitrio noſtro relinquitur, poſito breuitatis gratia $l(2+\sqrt{3}) = b$ capiatur abſciſſa $AX = \sqrt{(1+bb)}$, atque applicata imaginaria XY aequalis erit ipſi arcui quaeſito pariter imaginario, id quod eo magis notatu dignum videtur, quod iſte arcus eſt imaginarium tranſcendens.

§. 8. Primum igitur Theorema analyticum cuius demonſtratio planior, vel ſaltem magis directa, deſideratur, ſiquidem eius veritas quibusdam iam ſatis euicta videatur, hoc modo proponatur:

Theorema I.

Omnes plane quantitates imaginariae, quaecunque in calculo analytico occurrere poſſunt, ad hanc formam ſimpliciſſimam a + b $\sqrt{-1}$ *ita reuocari poſſunt, vt litterae* a *et* b

 quan-

quantitates reales denotent. Eius igitur demonstrationem sagacissimis Analystis imprimis commendare non dubito.

§. 9. Sequentia duo theoremata rectificationem linearum curuarum respiciunt, ideoque ad Geometriam sublimiorem sunt referenda. Cum enim iam pridem a celeb. *Hermanno* methodus geometrica sit reperta, innumerabiles curuas algebraicas inueniendi, quae vel sint rectificabiles, vel quarum rectificatio a data quacunque quadratura pendeat (quam methodum deinceps ad Analysin puram transtuli et plurimum locupletaui, ita vt peculiarem speciem Analyseos infinitorum constituere videatur); inde vtique infinitae curuae algebraicae exhiberi possunt, quarum rectificatio a quadratura circuli pendeat. Omnes autem, excepto circulo, ita comparatae deprehenduntur, vt earum arcus aggregato cuipiam ex quantitate algebraica et arcu circulari aequentur, quantitatem autem illam algebraicam nullo modo ad nihilum redigere liceat; vnde sequens theorema tanquam verum proponere non dubito, etiamsi eius demonstrationem exhibere nondum potuerim.

Theorema II.

Praeter circulum nulla datur curua algebraica, cuius singuli arcus per arcus circulares simpliciter exprimi queant.

§. 10. Hoc theorema igitur eo redit, vt demonstretur, nullam aequationem algebraicam inter binas coordinatas orthogonales x et y exhiberi posse, vt formula integralis $\int \surd (dx^2 + dy^2)$ aequetur arcui cuipiam circulari, cuius sinus vel cosinus sit functio quaepiam ipsarum x et y, solo casu excepto quo aequatio inter x et y circulum indicat.

dicat. Quod quo clarius intelligatur denotet ϕ angulum seu arcum quemcunque indefinitum in circulo cuius radius $=1$, ac ponatur $\int\sqrt{(dx^2+dy^2)}=a\phi$, ideoque $dx^2+dy^2=aad\phi^2$, fiatque $dx=apd\phi$ et $dy=aqd\phi$, atque necesse est vt sit $pp+qq=1$. Praeterea vero ambas formulas $apd\phi$ et $aqd\phi$ ita integrabiles esse oportet, vt earum integralia per solos sinus vel cosinus anguli ϕ exprimi queant, quod dico nullo modo fieri posse, nisi curua fuerit ipse circulus.

§. 11. His autem conditionibus manifesto satisfiet si capiatur $p=\sin.(n\phi+\alpha)$ et $q=\cos.(n\phi+\alpha)$, denotante α angulum quemcunque constantem, n vero numerum rationalem quemcunque; tum enim vtique erit $pp+qq=1$, et cum sit $dx=ad\phi\sin.(n\phi+\alpha)$ et $dy=ad\phi\cos.(n\phi+\alpha)$, hinc integrando elicitur $x=b-\frac{a}{n}\cos.(n\phi+\alpha)$ et $y=c+\frac{a}{n}\sin.(n\phi+\alpha)$, quae formulae, ob litteras α et n arbitrarias, innumeras curuas inuoluere videntur. Verum cum inde fiat $b-x=\frac{a}{n}\cos.(n\phi+\alpha)$ et $y-c=\frac{a}{n}\sin.(n\phi+\alpha)$, semper erit $(b-x)^2+(y-c)^2=\frac{aa}{nn}$, quae aequatio manifesto semper est pro circulo. Demonstrandum igitur est, pro conditionibus ante praescriptis loco litterarum p et q alios valores accipi non posse, qui iis satisfaciant.

§. 12. Cum autem nullum vestigium appareat ad talem demonstrationem perueniendi, videamus an per demonstrationem ad absurdum quicquam lucrari possit. Assumamus igitur praeter circulum aliam dari curuam algebraicam, cuius omnes arcus per arcus circulares metiri liceat. Sit igitur A Y ym talis curua algebraica, cuius quilibet arcus A Y ab initio A captus aequetur arcui cuipiam circulari, cuius sinus sit functio quaecunque algebraica abscissae A X, Tab. I. Fig. 5.

 ac

ac ſimili modo alius arcus quicunque A y etiam aequabitur arcui circulari, cuius ſinus erit ſimilis functio abſciſſae A x; hincque manifeſtum eſt, etiam differentiae horum arcuum, Y y, aequalem arcum circularem aſſignari poſſe, ita vt huius curuae omnes plane portiones Y y per ſimplices arcus circulares exprimi queant, ſicque demonſtrari oportebit, talem curuam algebraicam nullo prorſus modo exhiberi poſſe.

Tab. II. Fig. 6. §. 13. Primo hic autem obſeruo, ſi daretur talis curua, ea certe non in infinitum extendi poſſe, id quod ita oſtendo: Sit A$bcde$ etc. talis curua cum axe ABCDE in infinitum excurrens, in eaque accipiantur portiones aequales A b, bc, cd, de etc. quarum menſura ſit quadrans circuli, atque in applicatis B b, C c, D d etc. abſcindantur portiones B β, C γ, D δ, etc. quae ſint ſinibus arcuum A b, A c, A d etc. aequales, id quod etiam in ſingulis applicatis intermediis fieri intelligatur; ac manifeſtum eſt, ſingula haec puncta β, γ, δ etc. geometrice ſeu algebraice aſſignari poſſe, ita vt curua per omnia haec puncta ducta A $\beta\gamma\delta\varepsilon$ etc. futura eſſet algebraica. Quoniam vero ea habebit infinitas portiones alternatim ſupra et infra axem exiſtentes, ea ab axe ipſo in infinitis punctis interſecaretur, id quod in nulla curua algebraica locum habere poteſt. Vnde luculenter ſequitur, talem curuam A bcd in infinitum extenſam certe non dari poſſe; atque hinc iam eſt euictum, ſi darentur praeter circulum eiusmodi curuae algebraicae, quarum ſingulae portiones per arcus circulares menſurari queant, neceſſario eas in ſe redeuntes eſſe debere, tum enim abſurditas modo oſtenſa ceſſare poſſet, ita vt ſimili modo nihil abſurdi inde inferri poſſit.

§. 14.

§. 14. Sit igitur A B P Q R S talis curua algebraica in se rediens, cuius omnes plane portiones per arcus circulares metiri liceat, quae tamen non sit circulus; tum sumta quacunque portione A B, a quouis alio puncto P abscindi poterit portio P Q illi aequalis, quae tamen illi maxime erit dissimilis, quandoquidem curuamen, seu radius osculi, maxime differre potest in his portionibus, quales sunt A B, P Q, R S etc. Quanquam autem in hoc equidem nullam contradictionem ostendere possum, tamen demonstrari potest, si vnica talis curua daretur, ex ea infinitas alias inter se diuersas geometrice construi posse. Tum vero ex quolibet earum porro simili modo infinitas alias, ex earumque denuo qualibet infinitas alias, sicque in infinitum; ita vt multitudo talium curuarum satisfacientium foret non solum numerus infinitus, sed adeo potestas infinitesima infiniti. Quare cum adhuc nullo modo talis curua reperiri potuerit, nonne hinc iure concludere licebit, nullas plane dari huiusmodi curuas algebraicas? Tab. II. Fig. 7.

§. 15. Ad hoc autem demonstrandum, insignes illae proprietates, quas Vir celeberr. *Ioannes Bernoulli* de motu reptorio et curuis aequae amplis in lucem produxit, summo cum successu in vsum vocari poterunt, Fundamentum autem huius eximiae methodi in hoc consistit. Si habeantur duae curuae vtcunque diuersae aym et $a'y'm'$, in iisque capiantur arcus ay et $a'y'$ aeque ampli, ita vt ductis ad puncta y et y' normalibus yr et $y'r'$, quae axibus ab et $a'b'$ in r et r' occurrant, qui ipsi ad curuas normales supponuntur in a et a', anguli ary et $a'r'y'$ fiant inter se aequales, ex quo hi arcus ay et $a'y'$ aeque ampli sunt apellati; quibus positis, si hinc noua curua A Y M Tab. II. Fig. 8 et 9.

 ita

ita construatur, vt sumta abscissa $AX = m.\,ax + n\,a'x'$, constituatur applicata $XY = m.\,xy + n.\,x'y'$, tum etiam huius nouae curuae AM arcus AY erit $= m.\,ay + n.\,a'y'$. Quod si enim pro curuis datis ponamus abscissas $ax = x$ et $a'x' = x'$, applicatas vero $xy = y$ et $x'y' = y'$, erit subnormalis $xr = \frac{y\,dy}{dx}$ et $x'r' = \frac{y'\,dy'}{dx'}$, hincque tang. $ary = \frac{dx}{dy}$ et tang. $a'r'y' = \frac{dx'}{dy'}$. Quare cum hi anguli sint aequales, posito $\frac{dy}{dx} = p$, seu $dy = p\,dx$, erit etiam $\frac{dy'}{dx'} = p$, siue $dy' = p\,dx'$. Hinc igitur colligitur arcus $ay = \int dx\,\sqrt{(1+pp)}$ et arcus $a'y' = \int dx'\,\sqrt{(1+pp)}$. Iam in curua inde constructa AY erit abscissa $AX = X = mx + nx'$, applicata vero $XY = Y = my + ny'$, hincque $dX = m\,dx + n\,dx'$ et $dY = m\,dy + n\,dy' = p(m\,dx + n\,dx')$, ideoque erit $dY = p\,dX$ et arcus AY aeque amplus erit ac duo praecedentes ay et $a'y'$; hinc ergo huius nouae curuae arcus erit

Tab. II. Fig. 10.

$$AY = \int dX\sqrt{(1+pp)} = m\int dx\sqrt{(1+pp)} + n\int dx'\sqrt{(1+pp)},$$

vnde manifestum est fore arcum $AY = m.\,ay + n.\,a'y'$.

§. 16. Hoc iam fundamento stabilito, si ambae curuae ay ot $a'y'$ ita fuerint comparatae, vt arcus ay et $a'y'$ per arcus circulares mensurari queant, tum etiam curuae inde descriptae arcus AY etiam per arcum circularem mensurabitur, si modo litterae m et n denotent numeros rationales quoscunque. Ex quo iam intelligitur, ex illis curuis datis ay et $a'y'$ innumerabiles curuas AY eiusdem proprietatis construi posse. Hic autem obseruandum est, si ambae curuae datae ay et $a'y'$ fuerint circuli, curuam illam descriptam AY fore quoque circulum, cuius radius $RA = RY$ erit $= m.\,ra + n.\,r'a'$, ita vt hoc solo casu nulla noua curva resultet, id quod per se est perspicuum. Statim autem ac vel altera earum curuarum ay et $a'y'$, vel etiam ambae

non

non fuerint circuli, tum quoque curua descripta A Y certe non erit circulus, atque adeo in infinitum variari poterit, prouti numeris *m* et *n* alii atque alii valores tribuantur.

§. 17. Hinc ergo si pro curua *a y* accipiatur curva illa supra memorata, cuius scilicet singulos arcus per circulares mensurare posse assumimus, eamque a puncto quocunque A incipientem; pro altera autem *a' y'* circulum quemcunque, constructio modo tradita nobis suppeditabit innumerabiles curuas A Y eadem indole praeditas, vt arcui A Y aequalis arcus circularis assignari queat. Tum vero etiam, sumta curua *a y* aequali, ramo figurae illius a puncto A extenso, pro curua vero *a' y'* alius quicunque eiusdem curvae ramus ab alio puncto P protensus, hinc etiam innumerabiles aliae nouae curuae A Y describi poterunt, quae vtique omnes quoque erunt algebraicae; vnde manifestum est, si harum nouarum curuarum rami in locum alterius curuae datae *a y* vel etiam vtriusque substituantur; tum hoc modo infinita alia curuarum genera construi posse, quam multiplicationem adeo in infinitum augere licebit. Quare cum nulla adhuc eiusmodi curua a circulo diuersa erui potuerit, maxime verisimile est, ac fortasse tanquam rigide demonstratum spectari potest: nullam prorsus in rerum natura dari huiusmodi curuam algebraicam a circulo diuersam. Tab. II. Fig. 11.

§. 18. Quod hactenus de circulo est allatum etiam ad logarithmos extendi potest, quippe quos cum arcubus circularibus imaginariis comparare licet, vnde sequens theorema Geometris tanquam aeque certum et memoratu dignum ac praecedens commendare sustineo.

Theo-

Theorema III.

Nulla prorſus datur curua algebraica, cuius ſinguli arcus ſimpliciter per logarithmos exprimi queant. Ita vt hoc theorema nullam prorſus exceptionem, quemadmodum praecedens, poſtulet.

§. 19. Notum eſt rectificationem parabolae a logarithmis pendere, verum ſinguli eius arcus non per ſimplices logarithmos, ſed per aggregatum ex logarithmo et quapiam quantitate algebraica exprimuntur, ita vt hinc nulla exceptio theorematí inferatur. Hic autem primo obſeruandum eſt, vt ante, ſi talis daretur curua algebraica AYy, cuius omnes arcus in puncto A terminati per logarithmos aſſignari poſſent, vt verbi gratia eſſet $AY = alP$ et $Ay = alp$, ita vt P et p eſſent certae functiones algebraicae ambarum coordinatarum AX, XY et Ax, xy, tum etiam differentiam horum arcuum logarithmo exprimi poſſe, quandoquidem foret $Yy = al\frac{p}{P}$. Hinc ergo poſita abſciſſa $AX = x$ et applicata $XY = y$ demonſtrandam eſt, nullam dari aequationem algebraicam inter x et y, vt inde fiat $\int \surd(dx^2+dy^2) = alv$, denotante v functionem quampiam algebraicam ipſarum x et y; vnde ſi ponamus $dx = \frac{apdv}{v}$ et $dy = \frac{aqdv}{v}$, neceſſe eſt vt fiat $pp + qq = 1$. Praeterea vero requiritur vt ambae formulae $\int\frac{pdv}{v}$ et $\int\frac{qdv}{v}$ fiant algebraice integrabiles, cuius ergo impoſſibilitatem demonſtrari oportet.

Tab. I. Fig. 5.

§. 20. Quemadmodum mihi pro praecedente theoremate licuit oſtendere, nullam dari curuam in infinitum extenſam illi ſatisfacientem, ita hic ſimili modo oſtendi poteſt, nullam dari curuam in ſe redeuntem algebraicam, quae huic theorematí conueniat. Sit enim curua AYBDA curua in ſe se rediens, cuius omnes arcus AY per logarithmos exhiberi queant, ita vt in applicata XY, si opus est producta, algebraice assignari possit punctum Z, vt arcus AY fiat $= \log. XZ$; tum ergo, quia curua in se est rediens et arcui $AYBDAY$ eaedem coordinatae AX et XY conueniunt, aliud quoque dabitur punctum Z, cuius logarithmus huic arcui aequetur. Ac si circumferentia totius curuae ponatur $= c$, infinita talia spatia XZ, XZ', XZ'', XZ''', etc. assignari poterunt, quorum logarithmi aequentur arcubus AY, $AY + c$, $AY + 2c$, $AY + 3c$, et in genere $AY + nc$, denotante n numerum integrum quemcunque tam negatiuum quam positiuum; atque quia omnia haec puncta simili formula algebraica continebuntur, omnia quoque in eadem curua algebraica existerent, quae ergo a singulis applicatis XY productis in infinitis punctis secaretur, id quod naturae curuarum algebraicarum aduersatur.

§. 21. Quod si ergo daretur talis curua, cuius singulos arcus logarithmis metiri liceret, ea certe in infinitum excurreret. Iam vero ex vnica tali curua, ope propositionis fundamentalis, circa curuas aeque amplas supra allatae, pari modo, quo ibi processimus, infinities-infinita noua genera talium curuarum exhiberi possent; vnde cum nulla adhuc talis curua erui potuerit, si non prorsus certum, saltem maxime verisimile est, nullas plane dari eiusmodi curuas algebraicas.

§. 22. Ceterum si modo theorema secundum firmiter fuerit demonstratum, etiam huius demonstratio pro confecta esset habenda. Cum enim elementum arcus circuli, cuius radius $= a$ et sinus $= x$, sit $\frac{a\,dx}{\sqrt{(aa - xx)}}$, si radium ita imaginarium concipiamus, vt sit $a = c\sqrt{-1}$, elementum arcus eus fiet $\frac{c\,dx\sqrt{-1}}{\sqrt{(-cc-xx)}} = -\frac{c\,dx}{\sqrt{(cc+xx)}}$, quod ergo erit reale, etiamsi radius circuli sit imaginarius, eiusque adeo integrale erit $c\,l\frac{\sqrt{(cc+xx)}-x)}{c}$, vbi maxime mirum videri potest, quod arcus circuli imaginarii nihilo minus sint reales et quidem per logarithmos assignabiles. Atque hinc iam tuto concludere poterimus, quemadmodum praeter circulum nullae aliae dantur lineae curuae, cuius singulos arcus per circulares metiri liceat, ita etiam praeter circulum imaginarium nullas dari curuas algebraicas, quarum singulos arcus per logarithmos metiri liceat. Quoniam autem circulus imaginarius plane existere nequit, prorsus nullae curuae algebraicae exhiberi posse sunt censendae, quarum singulos arcus per logarithmos exprimere liceat.

DE RELATIONE INTER TERNAS PLVRESVE QVANTITATES INSTITVENDA.

§. 1.

Propositis duabus quantitatibus A et B, earum relatio, seu ratio definitur, dum quaeruntur duo numeri integri α et β, iique minimi, vt fiat $\alpha A = \beta B$; vnde si quantitates A et B fuerint inter se commensurabiles, istos numeros α et β semper accurate assignare licebit; sin autem sint incommensurabiles, numeros α et β ita dare licebit, vt discrimen inter formulas αA et βB sit minimum, vel ita paruum, vt propius ad aequalitatem inter has formulas αA et βB accedi nequeat, nisi pro α et β maiores numeri adhibeantur. Hocque modi solui solet problema olim a Wallisio propositum, quo, propositis duobus numeris quantumuis magnis A et B, rationes in minoribus numeris requiruntur, qui tam exacte eorum rationem exprimant, quam fieri potest numeris non maioribus adhibendis.

§. 2. Simili modo si tres proponantur quantitates A, B et C, reperiri poterunt tres numeri integri α, β et γ, vt fiat $\alpha A = \pm \beta B \mp \gamma C$; et quidem omnes possibiles valores pro his numeris α, β, γ assignare licebit, quibus inuentis haud difficile erit, minimos numeros pro α, β et γ exhibere, atque hoc modo relatio inter ternas quantitates propositas A, B et C planissime indicari videtur. Methodus autem hos tres numeros α, β, γ inuestigandi similis erit illi, qua relatio inter duas tantum quantitates definiri solet, et quae eiusmodi operationibus absoluitur, quibus maximus communis diuisor duorum numerorum indagari solet, id quod sequenti exemplo illustremus.

§. 3. Propositae igitur sint tres sequentes quantitates: $A = 49$, $B = 59$ et $C = 75$, et quaerantur numeri a, b, c, vt fiat $49a + 59b + 75c = 0$, vbi quidem a, b, c numeros integros, siue positiuos, siue negatiuos significent. Iam diuidatur aequatio illa per minimam propositarum quantitatum, scilicet per 49, et quoti ex posterioribus terminis oriundi resoluantur in partes integras et fractas, ac seorsim exhibeantur, quae quoniam iunctim sumtae nihilo debent aequari, partes integras statuamus aequales numero integro $+d$, fractae autem eidem numero negatiuo $-d$; hocque modo duae hinc nascentur aequationes

$$a + b + c = d \text{ et } \frac{10b + 26c}{49} = -d.$$

Iam ex postrema aequatione fit $10b + 26c + 49d = 0$, quae prorsus vt prima tractetur, scilicet per 10 diuisa dabit

$$b + 2c + 4d = +3e \text{ et } \frac{6c + 9d}{10} = -3e.$$

Hic scilicet, quia numeri 6 et 9 communem diuisorem habent 3, loco simplicis litterae e statim scripsimus $3e$, sicque noua aequatio erit $2c + 3d + 10e = 0$, quae, per 2 di-

2 diuisa, similique modo distributa, suppeditat has aequationes:

$$c + d + 5e = + f \text{ et } \frac{d}{2} = - f,$$

quae vltima statim dat $d = - 2f$, atque hic operationes terminantur, quoniam nullae amplius insunt fractiones.

§. 4. Cum igitur esse debeat $d = - 2f$, littera autem e non sit determinata, per has duas litteras e et f praecedentes sequenti modo regrediendo definientur:

$$c = 3f - 5e,\ b = 13e + 2f \text{ et } a = - 8e - 7f.$$

Solutio ergo generalis nostrae quaestionis, siue relatio inter ternos numeros propositos 49, 59, 75 sequenti aequatione continebitur:

$$-(8e + 7f)49 + (13e + 2f)59 + (3f - 5e)75 = 0$$

vbi pro e et f numeros quoscunque accipere licet.

§. 5. Videamus igitur, quales numeros pro e et f accipi conueniat, vt haec aequatio fiat simplicissima. Sumatur primo $f = 1$ et $e = - 1$, et relatio inuenta erit

$$1.49 - 11.59 + 8.75 = 0;$$

at si sumamus $e = 0$ et $f = - 1$, relatio erit

$$7.49 - 2.59 - 3.75 = 0,$$

quae sine dubio est simplicissima forma relationis. Atque ex hoc exemplo iam satis perspicuum est, quantumuis magnae fuerint quantitates A, B et C, quoniam continuo ad diuisores minores deuenimus, tandem omnes plane fractiones tolli, ac pro numeris a, b, c semper numeros integros obtineri.

§. 6. Cum igitur res ſit manifeſta, quando quantitates propoſitae A, B, C, D ſunt rationales, ſiue commenſurabiles inter ſe, etiam euidens eſt, ſi iſtae quantitates fuerint irrationales, vel adeo tranſcendentes, tum operationes hic vſitatas nunquam terminari, neque idcirco talem relationem exactam vllo modo exhiberi poſſe; veruntamen, quod his caſibus imprimis eſt notandum, ſi memoratae operationes alicubi abrumpantur, tum eiusmodi relationes eſſe prodituras, quae quidem rem non exacte, attamen vero proxime exhibeant, id quod ſaepenumero vſui eſſe poterit, quando inter huiusmodi quantitates relatio tantum proxime vera, et quidem in minimis deſideratur. Quomodo autem huiusmodi caſibus calculum tractare conueniat nonnullis exemplis oſtendemus.

§. 7. Sint igitur ternae quantitates $A = 1$, $B = \sqrt{2}$ et $C = \sqrt{3}$, ac primo, vt operationes ante adhibitae locum inuenire poſſint, has quantitates irrationales in fractiones decimales conuertamus, quas quidem non vltra ſextam notam continuemus. Eſt vero

$$\sqrt{2} = 1,414214 \text{ et } \sqrt{3} = 1,732051.$$

Iam per 1000000 multiplicando tota inueſtigatio ad numeros integros reuocetur, quandoquidem tota relatio in rationibus, quas hae quantitates inter ſe tenent ſubſiſtit, hocque modo aequatio principalis $a + b\sqrt{2} + c\sqrt{3} = 0$ transformabitur in hanc:

$$1000000.\,a + 1414214.\,b + 1732051.\,c = 0$$

quae diuiſa per 1000000 et vt ſupra in binas partes diſtributa dabit

$$a + b + c = +d \text{ et } \frac{414214.\,b + 732051.\,c}{1000000} = -d;$$

poſtrema

poſtrema igitur ad integros reducta praebet

$$414214.\,b + 732051.\,e + 1000000\,d = 0;$$

haecque aequatio per 414214 diuiſa eodemque modo tractata deducit ad has aequationes:

$$b + c + 2\,d = +\,e \text{ et } \frac{317837.\,c + 171572.\,d}{414214} = -\,e$$

quarum poſtrema reducitur ad hanc:

$$\frac{317837.\,c + 171572.\,d}{171572} + 414214.\,e = 0.$$

Tractetur iſta aequatio eodem modo, vt prodeant hae aequationes:

$$d + c + 2\,e = +\,f \text{ et } \frac{146255.\,c + 71070.\,e}{171572} = -\,f,$$

quarum poſtrema ad integros reducta ita ſe habet:

$$146265.\,c + 71070.\,e + 171572\,f = 0,$$

quae per 71070 diuiſa praebet

$$e + 2\,c + 2\,f = +\,g \text{ et } \frac{4125.\,c + 29432.\,f}{71070} = -\,g;$$

poſterior reducta fit

$$4125.\,c + 29432.\,f + 71070\,g = 0,$$

vnde per 4125 diuidendo ſe produnt hae duae aequationes:

$$c + 7\,f + 17\,g = +\,h \text{ et } \frac{575.\,f + 945.\,g}{4125} = -\,h,$$

at haec poſterior ad integros reducta praebet iſtam:

$$575.\,f + 945.\,g + 4125.\,h = 0.$$

Diuidatur nunc per 575 prodibitque

$$f + g + 7\,h = +\,i \text{ et } \frac{370.\,g + 100.\,h}{575} = -\,i$$

ſiue $\frac{74.\,g + 20.\,h}{115} = -\,i$, quae reducta fit

$$74.\,g + 20.\,h + 115\,i = 0,$$

vnde

vnde per 20 diuidendo hae oriuntur aequationes:

$$h + 3g + 5i = + k \text{ et } \frac{14.g + 15.i}{20} = - k.$$

§. 8. Hoc modo has operationes continuari liceret, quousque libuerit; verum quia fractiones decimales non vltra sextam figuram sunt productae, per has operationes vltimae numerorum nostrorum figurae continuo magis fiunt incertae, vnde in vltima aequatione binos numeros 14 et 15 tanquam aequales inter se spectare licebit, vnde capi poterit $g = 1$ et $i = -1$, fietque $k = 0$, atque hinc regrediendo sequentes valores reperientur:

$h = 2, f = -16, c = 97, e = -161, d = +209, b = -676, a = -788,$

sicque relatio quaesita ita se habebit:

$$788 - 676.\sqrt{2} + 97\sqrt{3} = 0, \text{ siue}$$

$$676\sqrt{2} - 97.\sqrt{3} = 788$$

cuius error vix vltra sextam figuram decimalem exsurget.

§. 9. Quantumuis autem haec relatio ad veritatem accedat: tamen inde neutiquam concludere licet, eam penitus veritati esse consentaneam. Si enim, denotantibus a, b, c numeros rationales, esset exacte $a = b\sqrt{2} + c\sqrt{3}$, tum sumtis quadratis foret $aa = 2bb + 3cc + 2bc\sqrt{6}$, hincque $\sqrt{6} = \frac{aa - 2bb - 3cc}{2bc}$, ideoque $\sqrt{6}$ foret numerus rationalis, quod vtique maxime esset absurdum; atque hoc idem etiam de omnibus aliis numeris radicalibus cuiuscunque ordinis est tenendum, ita vt quaelibet quantitas irrationalis natura sua tantopere discrepet ab omnibus aliis irrationalibus tam eiusdem quam diuersorum graduum, vt nulla plane relatio rationalis inter plures huiusmodi quantitates surdas diuersas locum habere possit.

§. 10.

§. 10. Vtrum autem quantitates transcendentes, veluti qui peripheriam circuli inuoluunt, siue logarithmi, etiam cum nullis quantitatibus radicalibus comparari queant, adhuc maxime incertum videtur, siquidem a nemine adhuc talis impossibilitas est ostensa. Tantum quidem satis euictum videtur, peripheriam π, circuli cuius diameter $= 1$, nullam comparationem cum formulis radicalibus quadraticis simplicibus admittere, quoniam aliter fractio continua ipsi π aequalis indices periodicos habere deberet, quod tamen neutiquam euenire videtur. Num autem quantitas π cum talibus formulis compositis nullo prorsus modo comparari queat, in dubio relinquere cogimur; quamobrem talem inuestigationem pro relatione quantitatum π, $\sqrt{2}$ et $\sqrt{3}$ methodo modo exposita suscipiamus.

§. 11. Euoluamus igitur modo explicato hanc aequationem:

$$a\sqrt{2} + b\sqrt{3} + c.\pi = 0,$$

quae in numeris integris proxime veris ita se habet:

$$1414214.a + 1732051.b + 3141593.c = 0,$$

quae per minimum numerum diuisa praebet has aequationes:

$$a + b + 2c = d \text{ et } \frac{317837.b + 313165.c}{1414214} = -d.$$

Postrema aequatio ergo in integris fit

$$317837.b + 313165.c + 1414214.d = 0,$$

quae iterum per minimum numerum diuidatur, quo facto produnt hae duae aequationes:

$$c + b + 4d = e \text{ et } \frac{4672.b + 161554.d}{313165} = -e,$$

quae posterior reducta fit

$$4672.b + 161554.d + 313165.e = 0.$$

 Diuiden-

Diuidendo per 4672 hae colliguntur aequationes:

$$b + 34\,d + 67\,e = f \text{ et } \frac{2706.\,d + 141.\,e}{4672} = -f.$$

§. 12. Operationes has vlterius non prosequor, quoniam, si exacta daretur relatio, ea sine dubio non adeo complicata esset futura. Prope veras autem tales relationes exhibuisse parum iuuaret. Vnde sententia satis certa videtur, quod peripheria circuli tam peculiare genus quantitatum transcendentium constituat, vt cum nullis aliis quantitatibus, siue surdis, siue alius generis transcendentibus, nullo modo se comparari patiatur.

§. 13. Infinita autem alia dantur transcendentium genera, quae neque ad circulum neque ad logarithmos reduci possunt, etiamsi quampiam affinitatem cum his quantitatibus tenere videantur; ac si forte tales quantitates cum hactenus cognitis exactam quandam relationem tenerent, quam directe ex principiis analyticis definire non licet, haec methodus vnicam viam suppeditare videtur, cuius beneficio huiusmodi relationes quasi diuinando explorare licebit.

§. 14. Huiusmodi igitur casum singularem, qui talem relationem non respuere videtur, hic accuratius euoluam, scilicet summam seriei reciprocae cuborum

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \text{ etc.},$$

quam nullo adhuc modo siue ad circulum siue ad logarithmos reducere potui, cum tamen summae potestatum parium omnes per potestates pares ipsius π exhiberi queant, summa autem primarum potestatum

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{ etc.}$$

loga-

logarithmum binarii exprimat.

§. 15. Cum igitur haec ſeries reciproca cuborum

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} + \text{etc.}$$

cubum huius ſeriei

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.}$$

in ſe complectatur, probabile videtur, in eius ſumma $(\log. 2)^3$ occurrere debere, neque tamen cuipiam multiplo huius quantitatis aequari certum eſt. Deinde vero, cum eadem ſeries in ſe complectatur productum ex binis praecedentibus, ſcilicet:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.} = l2 \text{ et}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = \frac{\pi\pi}{6}:$$

ſuſpicari licet, ibidem quoque productum $\frac{\pi\pi}{6} l2$ occurrere; quamobrem operae pretium erit inquirere, num forte ſumma ſeriei reciprocae cuborum tali formulae compoſitae:

$$\alpha (l2)^3 + \beta . \frac{\pi\pi}{6} . l2$$

aequetur, ita vt α et β ſint numeri rationales.

§. 16. Per approximationes autem olim ſummam ſeriei reciprocae cuborum ita aſſignaui:

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.} = 1,202056903,$$

vnde ſi eius pars quarta ſubtrahatur, prodit ſumma huius ſeriei:

$$1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \text{etc.} = 0,901542677$$

quam breuitatis gratia ponamus $= A$, et quaeramus numeros a, b, c, vt fiat

$$aA + b(l2)^3 + c\,l2 . \frac{\pi\pi}{6} = 0,$$

 vbi

vbi cum sit

$$\frac{\pi\pi}{6} = 1,644934066 \quad \text{et} \quad l\,2 = 0,693147180$$

colligimus fore proxime

$$(l\,2)^2 = 0,333025 \quad \text{et} \quad l\,2.\frac{\pi\pi}{6} = 1,140182$$

vnde relatio euoluenda erit

$$901543.a + 333025.b + 1140182.c = 0.$$

§. 17. Pro hac igitur aequatione operationes instituantur vt supra, eritque diuidendo per 333025

$$b + 2a + 3c = d \quad \text{et} \quad \frac{235493.a + 141107.c}{333025} = -d;$$

at posterior ad integros reducta praebet

$$235493.a + 141107.c + 333025.d = 0,$$

vnde diuidendo per 141107 deducimus has aequationes:

$$a + c + 2d = e \quad \text{et} \quad \frac{94386.a + 50811.d}{141107} = -e, \text{ siue}$$

$$94386.a + 50811.d + 141107.e = 0$$

qua aequatione diuisa per 50811 colligitur

$$a + d + 2e = f \quad \text{et} \quad \frac{43575.a + 39485.e}{50811} = -f;$$

at haec posterior reducta ad integros fit

$$43575.a + 39485.e + 50811.f = 0,$$

vnde porro diuidendo per minimum numerum oriuntur hae aequationes:

$$a + e + f = +g \quad \text{et} \quad \frac{4090.a + 11326.f}{39485} = -g, \text{ siue}$$

$$4090.a + 11326.f + 39485.g = 0,$$

vnde formentur hae aequationes:

$$a + 2f + 9g = +h \quad \text{et} \quad \frac{3146.f + 2675.g}{4090} = -h$$

etc. etc.

§. 18.

§. 18. Superfluum foret, has operationes vlterius continuare, quoniam hinc iam satis intelligere licet, nullam dari relationem tam concinnam inter ternas quantitates assumtas, vt veritati consentanea censeri posset. Cum igitur inuestigationem huius summae reciprocae cuborum tot variis modis frustra explorare tentassem, atque haec methodus etiam inutiliter sit in vsum vocata, merito de tali inuentione desperandum videtur.

DE RELATIONE FRACTIONVM TRANSCENDENTIVM IN INFINITAS FRACTIONES SIMPLICES.

§. 1.

Proposita fractione quacunque algebraica $\frac{P}{Q}$, cuius tam numerator P quam denominator Q sint functiones rationales integrae quantitatis z, iam pridem ostendi quomodo ea in fractiones simplices resolui possit, quarum denominatores aequentur factoribus simplicibus denominatoris Q, numeratores vero sint constantes, siquidem variabilis z in denominatore Q plures habeat dimensiones quam in numeratore P. Quin etiam ostendi, quemadmodum pro quolibet factore simplici denominatoris fractio simplex respondens reperiri queat, sine vllo respectu ad reliquos factores habito. Ita si constet, denominatorem Q factorem complecti simplicem $z - a$, fractio simplex inde nata, quae erit huius formae: $\frac{\alpha}{z-a}$, facillime hoc modo definitur. Statuatur $\frac{P}{Q} = \frac{\alpha}{z-a} + R$, vbi R complectatur omnes fractiones simplices ex reliquis oriendas. Multiplicetur vtrinque per $z - a$, vt fiat

$$\frac{P(z-a)}{Q} = \alpha + R(z-a),$$

et

et quia α eſt quantitas conſtans, ea ſemper eundem retinebit valorem, quicunque valor variabili z tribuatur; quamobrem fiat vbique $z = a$, vt reliquarum fractionum ſimplicium ratio ex calculo excedat, et habebitur $\alpha = \frac{P(z-a)}{Q}$; ſiquidem in hac formula fiat $z = a$, tum autem numerator $P(z-a)$ in nihilum abit; verum, quia $z - a$ eſt factor denominatoris Q, etiam denominator Q in nihilum abibit. Hinc igitur per regulam conſuetam loco numeratoris ac denominatoris ſua differentialia ſubſtituantur, quandoquidem etiamnunc erit $\frac{Pdz + (z-a)dP}{dQ} = \alpha$, ſiquidem hic vbique loco z ſcribatur a. Ponamus igitur hoc caſu $z = a$ fieri $P = A$ et $\frac{dQ}{dz} = C$, quae ergo quantitates A et C facilime inueniuntur, tum igitur prodibit numerator quaeſitus $\alpha = \frac{A}{C}$, ita vt fractio ſimplex ex denominatoris factore $z - a$ oriunda ſit $= \frac{A}{C}(z-a)$, ita vt non opus ſit reliquos factores denominatoris noſſe. Simili autem modo pro ſingulis reliquis factoribus fractiones ſimplices reſpondentes determinabuntur, quarum omnium ſumma aequabitur fractioni propoſitae $\frac{P}{Q}$, dummodo variabilis z pauciores habeat dimenſiones in numeratore P quam in denominatore Q.

§. 2. Haec igitur principia ſequentes pro denominatore Q eiusmodi aſſumamus functiones tranſcendentes, quas in infinitos factores ſimplices reſoluere liceat, id quod euenit, ſi eae infinitis caſibus nihilo aequales euadant. Praeterea vero neceſſe eſt vt omnes iſti factores inter ſe ſint inaequales, quandoquidem factores aequales peculiarem reſolutionem poſtulant. Imprimis autem requiritur, vt productum omnium talium factorum ipſam functionem Q penitus exhauriat, quoniam quandoque factores imaginarii ſe inter-

intermiscere possent. Veluti si sumatur $Q = \text{tang.}\, \phi$, ea vtique omnibus iisdem casibus euanescit, quibus haec functio: $\sin.\, \phi$, hincque ambae istae functiones eosdem factores simplices inuoluunt, etiamsi inter se neutiquam sint aequales. Deinde vero numeratorem P ita comparatum esse oportet vt cum denominatore Q nullos habeat factores communes. Imprimis autem cauendum est, ne quantitas variabilis in numeratore ad totidem vel plures dimensiones assurgat quam in denominatore. Cum autem ea in denominatore ad infinitas dimensiones assurgere sit censenda, istud incommodum non erit pertimescendum, quamdiu variabilis in numeratore tantum finito dimensionum numero continetur. Sin autem eius potestates etiam in infinitum ascendant, saepenumero difficile erit iudicare, num dimensionum numerus maior sit vel minor quam in denominatore. Interim tamen etiam his casibus fractio proposita $\frac{P}{Q}$ omnes continebit fractiones simplices, ad quas methodus nostra perducit. Verum euenire potest vt praeter eas etiam quasdam partes quasi integras inuoluat. His igitur praenotatis sequentes casus euoluamus.

I. Sumatur $Q = \sin.\, \Phi$, vt fractio resoluenda sit $\frac{P}{\sin.\, \Phi}$.

§. 3. Quoniam formula $\sin.\, \phi$, denotante π semiperipheriam circuli cuius radius $= 1$, seu angulum duobus rectis aequalem, omnibus his casibus euanescit:

$$\phi = 0,\ \phi = \pm \pi,\ \phi = \pm 2\pi,\ \phi = \pm 3\pi,\ \text{etc.}$$

et in genere $\phi = \pm i\pi$, eius factores numero infiniti erunt

$$\phi,\ (\phi \pm \pi),\ (\phi \pm 2\pi),\ (\phi \pm 3\pi)\ \text{et in genere}\ (\phi \pm i\pi).$$

Aliunde autem certum est, hanc formulam $\sin.\, \phi$ praeter istos factores nullos alios siue reales siue imaginarios inuoluere; cum

cum enim fit:

$$\sin.\varphi = \varphi - \frac{\Phi^3}{1.2.3} + \frac{\Phi^5}{1.2.3.4.5} - \text{etc.}$$

constat hanc seriem aequari huic producto infinito:

$$\varphi\left(1-\frac{\Phi}{\pi}\right)\left(1+\frac{\Phi}{\pi}\right)\left(1-\frac{\Phi}{2\pi}\right)\left(1+\frac{\Phi}{2\pi}\right)\left(1-\frac{\Phi}{3\pi}\right)\left(1+\frac{\Phi}{3\pi}\right) \text{ etc.}$$

§. 4. Consideremus igitur nostri denominatoris $Q = \sin.\varphi$ factorem quemcunque $\varphi + i\pi$, vbi i denotet omnes plane numeros integros, tam positiuos quam negatiuos, ciphra non excepta; sitque fractio partialis hinc oriunda $\frac{\alpha}{\Phi + i\pi}$. Ad eius numeratorem α inueniendum statuatur primo in numeratore P vbique $\varphi = i\pi$, sitque quantitas inde resultans $= A$; deinde cum sit $Q = \sin.\varphi$, erit $dQ = d\varphi \cos.\varphi$, siue $\frac{dQ}{d\Phi} = \cos.\varphi$, vbi loco φ itidem scribi oportet $i\pi$, vt obtineamus C, vnde patet fore $C = \cos.i\pi$, ita vt sit $C = \pm 1$, vbi signum $+$ valebit pro numeris paribus, signum vero $-$ pro imparibus numeris loco i assumtis. Hoc igitur modo numerator fractionis nostrae erit $\alpha = \pm A$ ipsaque fractio quaesita $\pm \frac{A}{\Phi - i\pi}$. Hinc autem vlterius progredi non licet, quamdiu numeratorem in genere spectamus, vnde eius loco plures valores determinatos accipiamus, singulosque in sequentibus exemplis euoluamus.

1°. Sit $P = 1$ et fractio proposita $\frac{1}{\sin.\Phi}$.

§. 5. Hic igitur semper erit $A = 1$ et fractio simplex quaecunque $= \frac{\pm 1}{\Phi - i\pi}$, vbi signum superius valet si i numerus par, inferius vero si impar. Substituamus igitur successiue pro i omnes eius valores ordine

$$0, +1, -1, +2, -2, +3, -3, +4, -4 \text{ etc.}$$

 et

et resolutio nostrae fractionis $\frac{1}{\sin.\phi}$ in fractiones simplices ita se habebit:

$$\frac{1}{\sin.\phi}=+\frac{1}{\phi}-\frac{1}{\phi-\pi}-\frac{1}{\phi+\pi}+\frac{1}{\phi-2\pi}+\frac{1}{\phi+2\pi}-\frac{1}{\phi-3\pi}-\frac{1}{\phi+3\pi}+\text{etc.}$$

quae in hanc formam reducatur:

$$\frac{1}{\sin.\phi}=\frac{1}{\phi}+\frac{1}{\pi-\phi}-\frac{1}{\pi+\phi}-\frac{1}{2\pi-\phi}+\frac{1}{2\pi+\phi}+\frac{1}{3\pi-\phi}-\frac{1}{3\pi+\phi}-\text{etc.}$$

Contrahantur post primum terminum bini sequentium, vt nanciscamur hanc seriem:

$$\frac{1}{\sin.\phi}=\frac{1}{\phi}+\frac{2\phi}{\pi\pi-\phi\phi}-\frac{2\phi}{4\pi\pi-\phi\phi}+\frac{2\phi}{9\pi\pi-\phi\phi}-\frac{2\phi}{16\pi\pi-\phi\phi}+\text{etc.}$$

vnde deducitur sequens series memoratu digna

$$\frac{1}{2\phi\sin.\phi}-\frac{1}{2\phi\phi}=\frac{1}{\pi\pi-\phi\phi}-\frac{1}{4\pi\pi-\phi\phi}+\frac{1}{9\pi\pi-\phi\phi}-\frac{1}{16\pi\pi-\phi\phi}\ \text{etc.}$$

§. 6. Has quidem series iam olim fusius sum prosecutus: interim tamen pro sequentibus casibus haud inutile erit sequentes transformationes hic repetere. Ponamus igitur primo $\phi=\lambda\pi$, vt littera π ex seriebus elidatur, atque hinc nanciscemur

$$\frac{\pi}{\sin.\lambda\pi}=\frac{1}{\lambda}-\frac{1}{\lambda-1}-\frac{1}{\lambda+1}+\frac{1}{\lambda-2}+\frac{1}{\lambda+2}-\frac{1}{\lambda-3}-\frac{1}{\lambda+3}+\text{etc. et}$$

$$\frac{\pi}{2\lambda\sin.\lambda\pi}-\frac{1}{2\lambda\lambda}=\frac{1}{1-\lambda\lambda}-\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}-\frac{1}{16-\lambda\lambda}+\text{etc.}$$

atque hinc per differentiationem, spectando λ tanquam quantitatem variabilem, infinitas alias series notatu dignissimas elicere poterimus. Ex priore scilicet nanciscemur

$$\frac{\pi\pi\cos.\lambda\pi}{\sin.\lambda\pi^2}=\frac{1}{\lambda\lambda}-\frac{1}{(\lambda-1)^2}-\frac{1}{(\lambda+1)^2}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}-\frac{1}{(\lambda-3)^2}-\text{etc.}$$

Hinc igitur sequitur, si $\lambda=\frac{1}{2}$ fore

$$0=1-1-\frac{1}{9}+\frac{1}{9}+\frac{1}{25}-\frac{1}{25}-\text{etc.}$$

quod quidem est manifestum. At si $\lambda=\frac{1}{3}$ erit

$$\frac{2\pi\pi}{27}=1-\frac{1}{4}-\frac{1}{16}+\frac{1}{25}+\frac{1}{49}-\frac{1}{64}-\frac{1}{100}+\frac{1}{121}+\frac{1}{169}-\text{etc.}$$

Si

Si $\lambda = \frac{1}{3}$, oritur ſeries praecedens.

Si $\lambda = \frac{1}{4}$, prodit haec ſummatio:

$$\frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{9} - \frac{1}{25} + \frac{1}{49} + \frac{1}{81} - \frac{1}{121} - \frac{1}{169} + \text{etc.}$$

Quod ſi denuo differentiemus obtinebitur ſequens ſummatio:

$$\frac{\pi^3}{\sin.\lambda\pi^3} - \frac{\pi^3}{2\sin.\lambda\pi} = \frac{1}{\lambda^3} - \frac{1}{(\lambda-1)^3} - \frac{1}{(\lambda+1)^3} + \frac{1}{(\lambda-2)^3} + \frac{1}{(\lambda+2)^3} - \text{etc.}$$

ſicque continuo vlterius progredi licet.

§. 7. Simili modo etiam alteram formam differentiemus, quae reducta praebet

$$\frac{1}{2\lambda^4} - \frac{\pi}{4\lambda^3\sin.\lambda\pi} - \frac{\pi\pi\cos.\lambda\pi}{4\lambda\lambda\sin.\lambda\pi^2} = \frac{1}{(1-\lambda\lambda)^2} - \frac{1}{(4-\lambda\lambda)^2} + \frac{1}{(9-\lambda\lambda)^2} - \frac{1}{(16-\lambda\lambda)^2} \text{ etc.}$$

Quod ſi nunc ſumamus $\lambda = \frac{1}{2}$, prodibit iſta ſummatio:

$$\frac{1}{2} - \frac{\pi}{8} = \frac{1}{3^2} - \frac{1}{15^2} + \frac{1}{35^2} - \frac{1}{63^2} + \frac{1}{99^2} - \text{etc.},$$

quae ſeries prorſus noua omnem attentionem meretur; neque autem opus eſt hinc nouam differentiationem inſtituere.

§. 8. Poſteriorem autem ſummationem

$$\frac{\pi}{2\lambda\sin.\lambda\pi} - \frac{1}{2\lambda\lambda} = \frac{1}{1-\lambda\lambda} - \frac{1}{4-\lambda\lambda} + \frac{1}{9-\lambda\lambda} - \frac{1}{16-\lambda\lambda} + \text{etc.}$$

accuratius perpendamus, ac primo quidem cum ea ſemper debeat eſſe vera, quicquid pro λ aſſumatur, ſumamus $\lambda = 0$. Quia autem hoc caſu membrum ſiniſtrum abit in $\infty - \infty$, tractetur λ vt quantitas quam minima, et cum ſit $\lambda\pi = \lambda\pi - \frac{1}{6}\lambda^3\pi^3$, iſtud membrum euadet

$$\frac{\pi}{2\lambda(\lambda\pi - \frac{1}{6}\lambda^3\pi^3)} - \frac{1}{2\lambda\lambda},$$

quae fractiones ad communem denominatorem perductae dant

$$\frac{1 - 1 + \frac{1}{6}\lambda\lambda\pi\pi}{2\lambda\lambda(1 - \frac{1}{6}\lambda\lambda\pi\pi)} = \frac{\pi\pi}{12 - 2\lambda\lambda\pi\pi}.$$

Nune

Nunc igitur facto $\lambda = 0$, eius factor erit $= \frac{\pi\pi}{12}$, series autem ipsa hoc casu euadet

$$1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \text{etc.}$$

cuius summam constat esse $\frac{\pi\pi}{12}$.

§. 9. Manifestum porro est, quoties pro λ accipiatur numerus integer, vnum terminum seriei, ideoque etiam ipsam seriem fieri infinitam, quod egregie conuenit cum summa inuenta, quandoquidem hoc casu fit $\sin. \lambda\pi = 0$. Atque hinc nata est ista quaestio: si ille terminus seriei in infinitum abiens ad sinistram partem transferatur, quanta futura sit reliquorum terminorum summa. Ponamus scilicet esse $\lambda = 1$, et primus seriei terminus euadet infinitus, qui ergo ad sinistram partem translatus dabit

$$\frac{\pi}{2\lambda \sin. \lambda\pi} - \frac{1}{2\lambda\lambda} - \frac{1}{1-\lambda\lambda} = -\frac{1}{3} + \frac{1}{8} - \frac{1}{15} + \frac{1}{24} - \frac{1}{35} + \frac{1}{48} - \text{etc.}$$

Nunc ad valorem huius seriei inuestigandum statuatur λ vnitati tantum proxime aequale, ponendo $\lambda = 1 - \omega$, eritque

$$\sin. \lambda\pi = \sin. (\pi - \pi\omega) = \sin. \pi\omega; \text{ est vero}$$

$$\sin. \pi\omega = \pi\omega - \tfrac{1}{6}\pi^3\omega^3,$$

quo valore substituto prodibit

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)} - \frac{1}{2(1-\omega)^2} - \frac{1}{2\omega - \omega\omega}.$$

Primum autem membrum

$$\frac{1}{2(1-\omega)\omega(1-\frac{1}{6}\pi^2\omega^2)}, \text{ ob}$$

$$\frac{1}{1-\omega} = 1 + \omega + \omega^2 \text{ et}$$

$$\frac{1}{1-\frac{1}{6}\pi^2\omega^2} = 1 + \tfrac{1}{6}\pi^2\omega^2,$$

negli-

negligendo poteſtates ipſius ω quadrato altiores, transmutatur in hanc formam:

$$\frac{1}{2\omega}(1+\omega+\omega^2+\tfrac{1}{6}\pi\pi\omega\omega);$$

tertium autem membrum

$$-\frac{1}{2\omega(1-\frac{1}{2}\omega)}, \text{ ob } \frac{1}{1-\frac{1}{2}\omega}=1+\tfrac{1}{2}\omega+\tfrac{1}{4}\omega\omega$$

abit in

$$-\frac{1}{2\omega}(1+\tfrac{1}{2}\omega+\tfrac{1}{4}\omega\omega),$$

vnde primum et tertium membrum ſimul faciunt

$$\frac{1}{2\omega}(\tfrac{1}{2}\omega+\tfrac{3}{4}\omega\omega+\tfrac{1}{6}\pi\pi\omega\omega)=\tfrac{1}{4}+\tfrac{3}{8}\omega+\tfrac{1}{12}\pi\pi\omega$$

qui valor poſito $\omega=0$ fit $=\frac{1}{4}$, vnde ſecundum membrum, quod erit $-\frac{1}{2}$, iunctum dabit totam ſummam quaeſitam $-\frac{1}{4}$, ita vt ſit mutatis ſignis

$$\tfrac{1}{4}=\tfrac{1}{3}-\tfrac{1}{8}+\tfrac{1}{15}-\tfrac{1}{24}+\tfrac{1}{35}-\tfrac{1}{48}+\tfrac{1}{63} \text{ etc.}$$

cuius ratio eſt manifeſta, cum ſit

$$\tfrac{1}{3}=\tfrac{1}{2}(1-\tfrac{1}{3});\ \tfrac{1}{8}=\tfrac{1}{2}(\tfrac{1}{2}-\tfrac{1}{4});\ \tfrac{1}{15}=\tfrac{1}{2}(\tfrac{1}{3}-\tfrac{1}{5});\ \tfrac{1}{24}=\tfrac{1}{2}(\tfrac{1}{4}-\tfrac{1}{6});\ \text{etc.}$$

his enim valoribus ſubſtitutis et ſublatis terminis ſe deſtruentibus fiet $\frac{1}{4}=\frac{1}{2}-\frac{1}{2.2}=\frac{1}{4}$.

§. 10. Circa eandem autem feriem quaeſtio magis ardua occurrit, qua quaeritur ſumma feriei, ſi $\lambda\lambda$ fuerit numerus negatiuus, ideoque λ quantitas imaginaria. Ponatur igitur $\lambda\lambda=-\mu\mu$, ſiue $\lambda=\mu\sqrt{-1}$, ac feries nihilo minus erit realis, ſcilicet

$$\frac{1}{1+\mu\mu}-\frac{1}{4+\mu\mu}+\frac{1}{9+\mu\mu}-\frac{1}{16+\mu\mu}+\frac{1}{25+\mu\mu}-\text{etc.}$$

cuius ergo ſumma erit

$$-\frac{\pi}{2\mu\sqrt{-1}\,\sin.\pi\mu\sqrt{-1}}+\frac{1}{2\mu\mu},$$

cuius

cuius ergo valor realis quaeritur, siquidem nullum est dubium quin seriei valor fiat realis.

§. 11. In doctrina angulorum ostendi solet esse

$$\text{sin.}\,\phi = \frac{e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}}}{2\sqrt{-1}}.$$

Fiat igitur $\phi = \mu\pi\sqrt{-1}$, eritque

$$\phi\sqrt{-1} = -\mu\pi \text{ et } -\phi\sqrt{-1} = +\mu\pi,$$

vnde concluditur

$$\text{sin.}\,\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}$$

vnde summa quaesita erit

$$\frac{\pi}{\mu(e^{-\mu\pi} - e^{+\mu\pi})} + \frac{1}{2\mu\mu}.$$

Erit igitur

$$\frac{1}{1+\mu\mu} - \frac{1}{4+\mu\mu} + \frac{1}{9+\mu\mu} - \frac{1}{16+\mu\mu} + \frac{1}{25+\mu\mu} - \text{etc.}$$

$$= \frac{1}{2\mu^2} - \frac{\pi}{\mu(e^{\mu\pi} - e^{-\mu\pi})}.$$

2°. Sit $P = \phi$ et fractio proposita $\frac{\phi}{\text{sin.}\,\phi}$.

§. 12. Hic ergo ob numeratorem $P = \phi$ factor denominatoris primus ϕ tollitur, quemadmodum etiam nostra resolutio numeratorem ipsi respondentem nihilo praebet aequalem. Hic igitur pro denominatore $\phi - i\pi$ fit numerator $\frac{i\pi}{\text{cos}.\, i\pi} = \pm i\pi$, vbi signum superius valet si i numerus par, inferius si impar. Quod si ergo fuerit $i = 2n$, fractio inde nata erit $\frac{2n\pi}{\phi - 2n\pi}$; at si $i = -2n$, fractio erit

— 2

$-\frac{2n\pi}{\phi+2n\pi}$, at si fuerit $i=2n-1$, fractio erit $-\frac{(2n-1)\pi}{\phi-(2n-1)\pi}$; denique ex $i=-2n-1$ oritur $\frac{(2n-1)\pi}{\phi+(2n-1)\pi}$, quocirca series inventa erit

$$\frac{\phi}{\sin.\phi}=-\frac{\pi}{\phi-\pi}+\frac{\pi}{\phi+\pi}+\frac{2\pi}{\phi-2\pi}-\frac{2\pi}{\phi+2\pi}-\frac{3\pi}{\phi-3\pi}+\frac{3\pi}{\phi+3\pi}-\frac{4\pi}{\phi+4\pi}+\text{etc.}$$

siue

$$\frac{\phi}{\sin.\phi}=\frac{\pi}{\pi-\phi}+\frac{\pi}{\pi+\phi}-\frac{2\pi}{2\pi-\phi}-\frac{2\pi}{2\pi+\phi}+\frac{3\pi}{3\pi-\phi}+\frac{3\pi}{3\pi+\phi}-\frac{4\pi}{4\pi-\phi}-\text{etc.}$$

vnde si bini termini in vnum contrahantur, erit

$$\frac{\phi}{\sin.\phi}=\frac{2\pi\pi}{\pi\pi-\phi\phi}-\frac{8\pi\pi}{4\pi\pi-\phi\phi}+\frac{18\pi\pi}{9\pi\pi-\phi\phi}-\frac{32\pi\pi}{16\pi\pi-\phi\phi}+\text{etc.}$$

quae per $2\pi\pi$ diuisa producit hanc summationem:

$$\frac{\phi}{2\pi\pi\sin.\phi}=\frac{1}{\pi\pi-\phi\phi}-\frac{4}{4\pi\pi-\phi\phi}+\frac{9}{9\pi\pi-\phi\phi}-\frac{16}{16\pi\pi-\phi\phi}+\text{etc.}$$

Ac si ponatur $\phi=\lambda\pi$, prodibit

$$\frac{\lambda\pi}{2\sin.\lambda\pi}=\frac{1}{1-\lambda\lambda}-\frac{4}{4-\lambda\lambda}+\frac{9}{9-\lambda\lambda}-\frac{16}{16-\lambda\lambda}+\text{etc.}$$

vnde si fuerit $\lambda\lambda=-\mu\mu$, seu $\lambda=\mu\sqrt{-1}$, ob

$$\sin.\mu\pi\sqrt{-1}=\frac{e^{-\mu\pi}-e^{+\mu\pi}}{2\sqrt{-1}},\text{ erit}$$

$$\frac{\mu\pi}{e^{\mu\pi}-e^{-\mu\pi}}=\frac{1}{1+\mu\mu}-\frac{1}{4+\mu\mu}+\frac{1}{9+\mu\mu}-\frac{1}{16+\mu\mu}+\text{etc.}$$

atque hinc per differentiationem infinitas alias summationes deducere licebit.

3°. Sit numerator $P=\phi^2$ et fractio $\frac{\phi\phi}{\sin.\phi}$.

§. 13. Pro denominatore igitur $\phi-i\pi$ numerator erit $\pm ii\pi\pi$, vbi signum superius valet pro i numero pari, inferius vero pro impari. Hinc si loco i successiue scribantur numeri

$$+1-1+2-2+3-3+\text{etc.}$$

series

feries refultans erit

$$\frac{\phi\phi}{\sin.\phi} = -\frac{\pi\pi}{\phi-\pi} - \frac{\pi\pi}{\phi+\pi} + \frac{4\pi\pi}{\phi-2\pi} + \frac{4\pi\pi}{\phi+2\pi} - \frac{9\pi\pi}{\phi-3\pi} - \frac{9\pi\pi}{\phi+3\pi} + \text{etc.}$$

fiue

$$\frac{\phi\phi}{\sin.\phi} = \frac{\pi\pi}{\pi-\phi} - \frac{\pi\pi}{\pi+\phi} - \frac{4\pi\pi}{2\pi-\phi} + \frac{4\pi\pi}{2\pi+\phi} + \frac{9\pi\pi}{3\pi-\phi} - \frac{9\pi\pi}{3\pi+\phi} - \text{etc.}$$

Contractis igitur binis terminis fiet

$$\frac{\phi\phi}{\sin.\phi} = \frac{2\pi\pi\phi}{\pi\pi-\phi\phi} - \frac{8\pi\pi\phi}{4\pi\pi-\phi\phi} + \frac{18\pi\pi\phi}{9\pi\pi-\phi\phi} - \frac{32\pi\pi\phi}{16\pi\pi-\phi\phi}, \text{ fiue}$$

$$\frac{\phi}{2\sin.\phi} = \frac{\pi\pi}{\pi\pi-\phi\phi} - \frac{4\pi\pi}{4\pi\pi-\phi\phi} + \frac{9\pi\pi}{9\pi\pi-\phi\phi} - \frac{16\pi\pi}{16\pi\pi-\phi\phi} + \text{etc.}$$

Quod fi nunc quilibet terminus huius feriei in duas partes difcerpatur, quarum prior femper eft 1, binae fequentes feries nafcentur:

$$\frac{\phi}{2\sin.\phi} = \begin{cases} +1 \quad -1 \quad +1 \quad -1 \quad +1 \quad -\text{etc.} \\ +\frac{\phi\phi}{\pi\pi-\phi\phi} - \frac{\phi\phi}{4\pi\pi-\phi\phi} + \frac{\phi\phi}{9\pi\pi-\phi\phi} - \frac{\phi\phi}{16\pi\pi-\phi\phi} + \frac{\phi\phi}{25\pi\pi-\phi\phi} - \text{etc.} \end{cases}$$

Notum autem eft, feriei

$$+1 - 1 + 1 - 1 + 1 - 1 \text{ etc.}$$

fummam effe $= \frac{1}{2}$, qua ad alteram partem translata et per $\phi\phi$ diuifa prodibit

$$\frac{1}{2\phi\sin.\phi} - \frac{1}{2\phi\phi} = \frac{1}{\pi\pi-\phi\phi} - \frac{1}{4\pi\pi-\phi\phi} + \frac{1}{9\pi\pi-\phi\phi} - \text{etc.}$$

quae prorfus conuenit cum ferie in §. 5 inuenta.

4°. Sit $P = \phi^\gamma$, denotante γ numerum imparem quemcunque pofitiuum, vt fractio propofita fit

$$\frac{\phi^\gamma}{\sin.\, i\,\phi}$$

§. 14. Cum igitur pro denominatore $\phi - i\pi$ fiat $A = i^\gamma \pi^\gamma$ et $C = \pm 1$, erit numerator $\pm i^\gamma \pi^\gamma$, vnde cum γ fit numerus impar, figna noftrorum terminorum eadem

dem lege procedent atque in casu $P = \phi$, vbi $\gamma = 1$; vnde series hinc nata erit

$$\frac{\phi^\gamma}{\sin.\phi} = \frac{\pi^\gamma}{\pi-\phi} + \frac{\pi^\gamma}{\pi+\phi} - \frac{2^\gamma\pi^\gamma}{2\pi-\phi} - \frac{2^\gamma\pi^\gamma}{2\pi+\phi} + \frac{3^\gamma\pi^\gamma}{3\pi-\phi} + \frac{3^\gamma\pi^\gamma}{3\pi+\phi} - \text{etc.}$$

quae per π^γ diuisa praebet

$$\frac{\phi^\gamma}{\pi^\gamma\sin.\phi} = \frac{1}{\pi-\phi} + \frac{1}{\pi+\phi} - \frac{2^\gamma}{2\pi-\phi} - \frac{2^\gamma}{2\pi+\phi} + \frac{3^\gamma}{3\pi-\phi} + \frac{3^\gamma}{3\pi+\phi} - \text{etc.}$$

et binis terminis contractis erit

$$\frac{\phi^\gamma}{2\pi^{\gamma+1}\sin.\phi} = \frac{1}{\pi\pi-\phi\phi} - \frac{2^\gamma}{4\pi\pi-\phi\phi} + \frac{3^\gamma}{9\pi\pi-\phi\phi} - \frac{4^\gamma}{16\pi\pi-\phi\phi} + \text{etc.}$$

Statuamus nunc $\phi = \lambda\pi$ eritque

$$\frac{\lambda^\gamma\pi}{2\sin.\phi\pi} = \frac{1}{1-\lambda\lambda} - \frac{2^\gamma}{4-\lambda\lambda} + \frac{3^\gamma}{9-\lambda\lambda} - \frac{4^\gamma}{16-\lambda\lambda} + \text{etc.}$$

§. 15. Hinc si fuerit $\lambda = \mu\sqrt{-1}$, erit vt ante

$$\sin.\mu\pi\sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}.$$

Pro valore autem potestatis λ^γ duos casus euolui oportet, prouti fuerit $\gamma = 4n+1$ vel $\gamma = 4n-1$. Priore casu erit

$$\lambda^\gamma = (\mu\sqrt{-1})^{4n+1} = (\mu\sqrt{-1})^{4n} \times \mu\sqrt{-1}.$$

 Est

Est vero $(\mu\sqrt{-1})^{4n} = \mu^{4n}$, vnde erit $\lambda^{\gamma} = \mu^{4n+1}\sqrt{-1}$, hincque prodit sequens summatio realis:

$$\frac{\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1^{4n+1}}{1+\mu\mu} - \frac{2^{4n+1}}{4+\mu\mu} + \frac{3^{4n+1}}{9+\mu\mu} - \frac{4^{4n+1}}{16+\mu\mu} + \text{etc.}$$

Altero autem casu, quo $\gamma = 4n-1$, prius membrum capi debet negatiue, eritque

$$-\frac{\mu^{4n+1}\pi}{e^{\mu\pi}-e^{-\mu\pi}} = \frac{1}{1+\mu\mu} - \frac{2^{4n-1}}{4+\mu\mu} + \frac{4^{4n-1}}{9+\mu\mu} - \text{etc.}$$

Has autem summationes facile patet veras esse non posse, nisi γ sit numerus integer impar et quidem positiuus.

5. Sit numerator $P = \varphi^{\delta}$, denotante δ numerum parem positiuum quemcunque, et fractio $\frac{\varphi^{\delta}}{\sin.\varphi}$.

§. 15. Pro denominatore ergo $\varphi - i\pi$ numerator erit $\pm i^{\delta}\pi^{\delta}$, ambiguitate signorum eandem legem tenente. Hoc igitur casu ratio signorum perinde se habebit ac casu $P = \varphi\varphi$, eritque idcirco.

$$\frac{\varphi^{\delta}}{\sin.\varphi} = \frac{\pi^{\delta}}{\pi-\varphi} - \frac{\pi^{\delta}}{\pi+\varphi} - \frac{2^{\delta}\pi^{\delta}}{2\pi-\varphi} + \frac{2^{\delta}\pi^{\delta}}{2\pi+\varphi} + \frac{3^{\delta}\pi^{\delta}}{3\pi-\varphi} - \frac{3^{\delta}\pi^{\delta}}{3\pi+\varphi} + \text{etc.}$$

Quare si ponamus $\varphi = \lambda\pi$ erit haec series

$$\frac{\lambda^{\delta}\pi}{\sin.\lambda\pi} = \frac{1}{1-\lambda} - \frac{1}{1+\lambda} - \frac{2^{\delta}}{2-\lambda} + \frac{2^{\delta}}{2+\lambda} + \frac{3^{\delta}}{3-\lambda} - \frac{3^{\delta}}{3+\lambda} - \text{etc.}$$

hinc binis terminis in vnum contrahendis fiet

$$\frac{\lambda^{\delta-1}\pi}{2\sin.\lambda\pi} = \frac{1^{\delta}}{1-\lambda\lambda} - \frac{2^{\delta}}{4-\lambda\lambda} + \frac{3^{\delta}}{3-\lambda\lambda} - \frac{4^{\delta}}{4-\lambda\lambda} + \text{etc.}$$

§. 16.

§. 16. Statuamus nunc etiam $\lambda = \mu\sqrt{-1}$, vt sit

$$\sin.\lambda\pi = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}}.$$

Pro valore autem ipsius $\lambda^{\delta-1}$ duos iterum casus euolui oportet, prout fuerit vel $\delta = 4n$, vel $\delta = 4n+2$. Priore casu, quo $\delta = 4n$, erit $\lambda^{4n} = \mu^{4n}$, ideoque $\lambda^{4n-1} = \frac{\mu^{4n-1}}{\sqrt{-1}}$; atque hinc orietur ista summatio:

$$\frac{-\mu^{4n-1}\pi}{e^{\mu\pi} - e^{-\mu\pi}} = \frac{1^{4n}}{1+\mu\mu} - \frac{2^{4n}}{4+\mu\mu} + \frac{3^{4n}}{9+\mu\mu} - \frac{4^{4n}}{16+\mu\mu} + \text{etc.}$$

Pro altero autem casu $\delta = 4n+2$ summatio ita se habebit.

$$\frac{+\mu^{4n+1}\pi}{e^{\mu\pi} - e^{-\mu\pi}} = \frac{1^{4n+2}}{1+\mu\mu} - \frac{2^{4n+2}}{4+\mu\mu} + \frac{3^{4n+2}}{9+\mu\mu} - \text{etc.}$$

§. 17. Hae autem summationes eatenus tantum veritati erunt consentaneae, quatenus pro exponentibus γ et δ numeri integri, prouti sunt definiti, accipiantur, nihilque impedit quo minus quantumuis magni assumantur. Cum enim denominator

$$\sin.\phi = \phi - \tfrac{1}{6}\phi^3 + \tfrac{1}{120}\phi^5 - \text{etc.}$$

ad dimensiones infinitas ipsius ϕ assurgat, dummodo maxima potestas in numeratore non fiat infinita, resolutio in fractiones semper ad veritatem perducit. Sin autem exponentes illi non essent integri positiui, sed fracti, vel adeo negatiui, resolutio in fractiones partiales locum plane habere nequit. Quamobrem si loco numeratoris P eiusmodi functiones ipsius ϕ statuamus, quae etiam ad infinitum dimensionum numerum adsurgant, tum de summa inuenta non amplius erimus

mus certi. Verum fieri poteſt, vt ad fractiones partiales inuentas inſuper quaedam partes integrae adiici debeant. Huiusmodi igitur caſus aliquos euoluamus.

6°. Sit numerator $P = \cos.\phi$ et fractio $= \frac{\cos.\phi}{\sin.\phi}$.

§. 18. Cum ſit

$$\cos.\phi = 1 - \tfrac{1}{2}\phi\phi + \tfrac{1}{24}\phi^4 - \tfrac{1}{720}\phi^6 + \text{etc.}$$

poteſtates ipſius ϕ in numeratore aeque in infinitum exſurgunt atque in denominatore; vnde fieri poſſet, vt haec fractio partem integram inuolueret, quae cum reperiatur ſi ſumatur $\phi = \infty$, foret iſta pars integra $= \frac{\cos.\infty}{\sin.\infty} = \cot.\infty$, quae autem in ſe prorſus eſt indeterminata. Interim tamen, quia totidem caſibus euadere poteſt negatiua atque poſitiua, medium ſumendo valor recte videri poteſt $= 0$; ceterum dubium per ſequentem euolutionem tolletur. Cum pro denominatore $\phi - i\pi$ fiat $A = \cos.i\pi$ et $C = \cos.i\pi$, erit numerator huius fractionis $= 1$; hinc ergo naſcetur ſequens ſeries:

$$\frac{\cos.\phi}{\sin.\phi} = \frac{1}{\phi} + \frac{1}{\phi-\pi} + \frac{1}{\phi+\pi} + \frac{1}{\phi-2\pi} + \frac{1}{\phi+2\pi} + \text{etc. ſiue}$$

$$\text{cotag.}\,\phi = \frac{1}{\phi} - \frac{1}{\pi-\phi} + \frac{1}{\pi+\phi} - \frac{1}{2\pi-\phi} + \frac{1}{2\pi+\phi} - \text{etc.}$$

Poſito igitur $\phi = \lambda\pi$, haec ſeries induet hanc formam:

$$\pi\cot.\lambda\pi = \frac{1}{\lambda} - \frac{1}{1-\lambda} + \frac{1}{1+\lambda} - \frac{1}{2-\lambda} + \frac{1}{2+\lambda} - \frac{1}{3-\lambda} + \text{etc.}$$

quae ſummatio an vera ſit per caſus inueſtigemus. Ac primo quidem ſi λ denotet numerum integrum, veritas confirmatur; ſemper enim aliquis ſeriei terminus fit infinitus; ſumma vero quoque fit infinita. Sumamus autem $\lambda = \frac{1}{2}$, erit $\pi\cot.\frac{\pi}{2} = 0$, ipſa autem ſeries prodit

$$\tfrac{2}{1} - \tfrac{2}{1} + \tfrac{2}{3} - \tfrac{2}{3} + \tfrac{2}{5} - \tfrac{2}{5} + \text{etc.}$$

vbi

vbi omnes termini ſe manifeſto deſtruunt. Sumamus autem inſuper $\lambda = \frac{1}{4}$, prodibitque

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \text{etc.}$$

quae eſt ſeries notiſſima Leibniziana. Sicque omne dubium circa veritatem huius ſummationis euaneſcit.

§. 19. Contrahamus binos terminos, primo excepto, in ſingulos, et obtinebimus

$$\pi \cot. \lambda \pi = \frac{1}{\lambda} - \frac{2\lambda}{1 - \lambda\lambda} - \frac{2\lambda}{4 - \lambda\lambda} - \frac{2\lambda}{9 - \lambda\lambda} - \frac{2\lambda}{16 - \lambda\lambda} - \text{etc.}$$

quae ſeries reducitur ad hanc formam:

$$\frac{1}{2\lambda\lambda} - \frac{\pi \cot. \lambda \pi}{2\lambda} = \frac{1}{1 - \lambda\lambda} + \frac{1}{4 - \lambda\lambda} + \frac{1}{9 - \lambda\lambda} + \frac{1}{16 - \lambda\lambda} + \text{etc.}$$

Quod ſi hic iterum ſtatuamus $\lambda = \mu \sqrt{-1}$, ob

$$\operatorname{cof.} \mu \pi \sqrt{-1} = \frac{e^{-\mu\pi} + e^{\mu\pi}}{2} \text{ et}$$

$$\operatorname{ſin.} \mu \pi \sqrt{-1} = \frac{e^{-\mu\pi} - e^{+\mu\pi}}{2\sqrt{-1}},$$

haec obtinebitur ſummatio:

$$-\frac{1}{2\mu\mu} + \frac{\pi(e^{+\mu\pi} + e^{-\mu\pi})}{(e^{\mu\pi} - e^{-\mu\pi})2\mu} = \frac{1}{1 + \mu\mu} + \frac{1}{4 + \mu\mu} + \frac{1}{9 + \mu\mu}$$
$$+ \frac{1}{16 + \mu\mu} + \text{etc.}$$

Nunc autem per ſe eſt manifeſtum, per differentiationem ſimili modo vt ſupra infinitas alias ſummationes obtineri poſſe.

II. Sumatur $Q = \cos.\zeta - \cos.\phi$, vt fractio resoluenda sit $\frac{P}{\cos.\zeta - \cos.\phi}$

§. 20. Cum sit denominator $Q = \cos.\zeta - \cos.\phi$, vbi angulus ζ vt datus et constans spectatur, is sequentibus casibus euanescit,

$$\phi = \pm\zeta;\ \phi = \pm 2\pi \pm \zeta;\ \phi = \pm 4\pi \pm \zeta;$$
$$\phi = \pm 6\pi \pm \zeta;\ \phi = \pm 8\pi \pm \zeta;\ \text{etc.}$$

ideoque in genere $\phi = \pm i\pi \pm \zeta$, vbi i denotat omnes numeros pares tam negatiuos quam positiuos; vnde denominatores fractionum simplicium quas quaerimus erunt

$$\phi-\zeta,\ \phi+\zeta,\ \phi-2\pi-\zeta,\ \phi-2\pi+\zeta\ \ \phi+2\pi-\zeta,\ \phi+2\pi+\zeta,\ \text{etc.}$$

hocque modo omnes fractiones simplices reperiemus, quarum omnium summa aequalis esse debet fractioni propositae $\frac{P}{\cos.\zeta - \cos.\phi}$.

§. 21. Consideremus nunc primo denominatorem simplicem in genere $\phi - i\pi - \zeta$, ac posito $\phi = i\pi + \zeta$ abeat numerator P in A. Deinde cum ex denominatore fiat $\frac{dQ}{d\phi} = \sin.\phi$, erit $C = \sin.(i\pi + \zeta) = \sin.\zeta$, vnde numerator huius fractionis erit $\frac{A}{C} = \frac{A}{\sin.\zeta}$, ideoque fractio hinc nata $\frac{A}{\sin.\zeta(\phi - i\pi - \zeta)}$. At vero pro denominatore $\phi - i\pi + \zeta$, si in numeratore P ponatur $\phi = i\pi - \zeta$, prodit quantitas B; ex denominatore autem fiet

$$C = \sin.(i\pi - \zeta) = -\sin.\zeta,$$

vnde orietur ista fractio: $-\frac{B}{\sin.\zeta(\phi - i\pi + \zeta)}$. Nunc igitur tantum opus est vt loco i successiue omnes numeri pares tam positiui qnam negatiui substituantur.

1°. Sit numerator $P = 1$ et fractio proposita

$$\frac{1}{cos.\,\zeta - cos.\,\Phi}.$$

§. 22. Pro binis igitur formulis generalibus erit tam $A = 1$ quam $B = 1$, vnde istae fractiones generales erunt

$$\frac{1}{sin.\,\zeta(\Phi - i\pi - \zeta)} - \frac{1}{sin.\,\zeta(\Phi - i\pi + \zeta)} = \frac{2\zeta}{sin.\,\zeta((\Phi - i\pi)^2 - \zeta\zeta)},$$

consequenter hinc deducemus sequentem summationem:

$$\frac{1}{cos.\,\zeta - cos.\,\Phi} = \frac{2\zeta}{sin.\,\zeta(\Phi\Phi - \zeta\zeta)} + \frac{2\zeta}{sin.\,\zeta((\Phi - 2\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{sin.\,\zeta((\Phi + 2\pi)^2 - \zeta\zeta)}$$
$$+ \frac{2\zeta}{sin.\,\zeta((\Phi - 4\pi)^2 - \zeta\zeta)} + \frac{2\zeta}{sin.\,\zeta((\Phi + 4\pi)^2 - \zeta\zeta)} + \text{etc.}$$

siue habebimus

$$\frac{sin.\,\zeta}{2\zeta(cos.\,\zeta - cos.\,\Phi)} = \frac{1}{\Phi\Phi - \zeta\zeta} + \frac{1}{(\Phi - 2\pi)^2 - \zeta\zeta} + \frac{1}{(\Phi + 2\pi)^2 - \zeta\zeta}$$
$$+ \frac{1}{(\Phi - 4\pi)^2 - \zeta\zeta} + \frac{1}{(\Phi + 4\pi)^2 - \zeta\zeta} + \text{etc.}$$

§. 23. Quod si ergo fuerit $\zeta = 0$, erit

$$\frac{1}{2 - 2\,cos.\,\Phi} = \frac{1}{\Phi^2} + \frac{1}{(\Phi - 2\pi)^2} + \frac{1}{(\Phi + 2\pi)^2} + \frac{1}{(\Phi - 4\pi)^2} + \frac{1}{(\Phi + 4\pi)^2} + \text{etc.}$$

Sit nunc porro $\Phi = \frac{\pi}{2}$, erit haec summatio:

$$\frac{\pi\pi}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \text{etc.}$$

vti quidem satis constat. Ponamus porro $\Phi = \pi$, prodibitque haec summatio:

$$\frac{\pi\pi}{4} = 1 + 1 + \frac{1}{9} + \frac{1}{9} + \frac{1}{25} + \frac{1}{25} + \frac{1}{49} + \frac{1}{49} + \text{etc.}$$

quae series cum praecedente congruit. Sin autem ponatur $\Phi = \lambda\pi$, erit

$$\frac{\pi\pi}{2(1 - cos.\,\lambda\pi)} = \frac{1}{\lambda\lambda} + \frac{1}{(\lambda - 2)^2} + \frac{1}{(\lambda + 2)^2} + \frac{1}{(\lambda - 4)^2} + \frac{1}{(\lambda + 4)^2} + \text{etc.}$$

quae summa etiam est $\frac{\pi\pi}{4(sin.\,\frac{1}{2}\lambda\pi)^2}$.

§. 24.

§. 24. Ponamus autem in genere $\zeta = \alpha\pi$ et $\Phi = \lambda\pi$, vt obtineatur ista summatio:

$$\frac{\pi \sin. \alpha\pi}{2\alpha(\cos. \alpha\pi - \cos. \lambda\pi)} = \frac{1}{\lambda\lambda - \alpha\alpha} + \frac{1}{(\lambda-2)^2 - \alpha\alpha} + \frac{1}{(\lambda+2)^2 - \alpha\alpha}$$
$$+ \frac{1}{(\lambda-4)^2 - \alpha\alpha} + \frac{1}{(\lambda+4)^2 - \alpha\alpha} + \text{etc.}$$

Quod si iam α fuerit quantitas imaginaria, siue $\alpha = \beta\sqrt{-1}$, summatio haec erit

$$\frac{\pi(e^{\beta\pi} - e^{-\beta\pi})}{2\alpha((e^{\beta\pi} + e^{-\beta\pi}) - 2\cos. \lambda\pi)} = \frac{1}{\lambda\lambda + \beta\beta} + \frac{1}{(\lambda-2)^2 + \beta\beta} + \frac{1}{(\lambda+2)^2 + \beta\beta}$$
$$+ \frac{1}{(\lambda-4)^2 + \beta\beta} + \frac{1}{(\lambda+4)^2 + \beta\beta} + \text{etc.}$$

§. 25. Hinc si proponatur haec fractio in seriem resoluenda: $\frac{1}{a - \cos. \Phi}$, siue $\frac{1}{a - \cos. \lambda\pi}$, duos casus considerari oportet, prouti a fuerit vel vnitate minor vel maior. Sit $a < 1$, vt fieri queat $a = \cos. \alpha\pi$, vnde fit $\alpha = \frac{A \cos. a}{\pi}$, atque invento α reperietur

$$\frac{1}{a - \cos. \lambda\pi} = \frac{2\alpha}{\pi\sqrt{(1-aa)}}\left(\frac{1}{\lambda\lambda - \alpha\alpha} + \frac{1}{(\lambda-2)^2 - \alpha\alpha} + \frac{1}{(\lambda+2)^2 - \alpha\alpha} + \frac{1}{(\lambda-4)^2 - \alpha\alpha}\right) + \text{etc.})$$

Sin autem fuerit $a > 1$, quaeri debet β, vt fiat

$$\frac{e^{\beta\pi} + e^{-\beta\pi}}{2} = a.$$

Hinc ergo fiet $e^{+2\beta\pi} + 1 = 2ae^{\beta\pi}$, vnde radice extracta reperitur $e^{\beta\pi} = a + \sqrt{(aa-1)}$ hincque $e^{-\beta\pi} = a - \sqrt{(aa-1)}$, vnde porro fiet

$$\beta\pi = l(a + \sqrt{(aa-1)}), \text{ ergo}$$
$$\beta = \tfrac{1}{\pi}l(a + \sqrt{(aa-1)}).$$

Inuento igitur hoc numero β postrema formula praebet hanc seriem:

$$\frac{\pi\sqrt{(aa-1)}}{2\beta(a - \cos. \lambda\pi)} = \frac{1}{\lambda\lambda + \beta\beta} + \frac{1}{(\lambda-2)^2 + \beta\beta} + \frac{1}{(\lambda+2)^2 + \beta\beta} + \frac{1}{(\lambda-4)^2 + \beta\beta} + \text{etc.}$$

conse-

consequenter habebimus

$$\frac{1}{a-\cos.\lambda\pi}=\frac{2\beta}{\pi\sqrt{(aa-1)}}\left(\frac{1}{\lambda\lambda+\beta\beta}+\frac{1}{(\lambda-2)^2+\beta\beta}+\frac{1}{(\lambda+2)^2+\beta\beta}+\frac{1}{(\lambda-4)^2+\beta\beta}+\text{etc.}\right)$$

casu autem medio, quo $a=1$, fit $\alpha=0$; tum vero ponatur $a=1-\infty$, eritque

$$\text{A}\cos.(1-\omega)=\text{A}\sin.\sqrt{(2\omega-\omega\omega)}=\sqrt{(2\omega-\omega\omega)}.$$

Est vero etiam

$$\sqrt{(1-aa)}=\sqrt{(2\omega-\omega\omega)},$$

vnde pro hoc casu seriei summatio erit

$$\frac{1}{1-\cos.\lambda\pi}=\frac{2}{\pi\pi}\left(\frac{1}{\lambda\lambda}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}+\frac{1}{(\lambda-4)^2}+\frac{1}{(\lambda+4)^2}+\text{etc.}\right)$$

Cum igitur sit

$$1-\cos.\lambda\pi=2\sin.\tfrac{1}{2}\lambda\pi^2,$$

habebimus hanc summationem:

$$\frac{\pi\pi}{4\sin.\frac{1}{2}\lambda\pi^2}=\frac{1}{\lambda\lambda}+\frac{1}{(\lambda-2)^2}+\frac{1}{(\lambda+2)^2}+\frac{1}{(\lambda-4)^2}+\text{etc.}$$

quae series iam supra § 23 est inuenta.

2°. Sit nunc $P=\sin.\Phi$ et fractio proposita $\frac{\sin.\Phi}{\cos\zeta-\cos.\Phi}$.

§. 26. Cum igitur sit $P=\sin.\Phi$ sumto, $\Phi=i\pi+\zeta$ erit $A=\sin.(i\pi+\zeta)=\sin.\zeta$; at posito $\Phi=i\pi-\zeta$ prodit $B=-\sin.\zeta$; hinc binae fractiones inde resultantes erunt

$$\frac{1}{\Phi-i\pi-\zeta}+\frac{1}{\Phi-i\pi+\zeta}=\frac{2\Phi-2i\pi}{(\Phi-i\pi)^2-\zeta\zeta}.$$

Quare si loco i successiue omnes eius valores scribamus, nanciscemur sequentem seriem:

$$\frac{\sin\Phi}{\cos.\zeta-\cos.\Phi}=\frac{2\Phi}{\Phi\Phi-\zeta\zeta}+\frac{2(\Phi-2\pi)}{(\Phi-2\pi)^2-\zeta\zeta}+\frac{2(\Phi+2\pi)}{(\Phi+2\pi)^2-\zeta\zeta}+\frac{2(\Phi-4\pi)}{(\Phi-4\pi)^2-\zeta\zeta}+\text{ etc.}$$

 siue

siue

$$\frac{\sin.\Phi}{2(\cos.\zeta-\cos.\Phi)}=\frac{\Phi}{\Phi\Phi-\zeta\zeta}+\frac{\Phi-2\pi}{(\Phi-2\pi)^2-\zeta\zeta}+\frac{\Phi+2\pi}{(\Phi+2\pi)^2-\zeta\zeta}+\frac{\Phi-4\pi}{(\Phi-4\pi)^2-\zeta\zeta}+\text{etc.}$$

§. 27. Hinc si fuerit $\zeta=0$, erit

$$\frac{\sin.\Phi}{2(1-\cos.\Phi)}=\frac{1}{\Phi}+\frac{1}{\Phi-2\pi}+\frac{1}{\Phi+2\pi}+\frac{1}{\Phi-4\pi}+\frac{1}{\Phi+4\pi}+\text{etc.}$$

cuius igitur seriei summa est $\frac{1}{2}\cot.\frac{1}{2}\Phi$. Hinc si ponamus $\Phi=\lambda\pi$, erit

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi=\frac{1}{\lambda}+\frac{1}{\lambda-2}+\frac{1}{\lambda+2}+\frac{1}{\lambda-4}+\frac{1}{\lambda+4}+\text{etc.}$$

et contrahendis binis terminis

$$\tfrac{1}{2}\pi\cot.\tfrac{1}{2}\lambda\pi=\frac{1}{\lambda}+\frac{2\lambda}{\lambda\lambda-4}+\frac{2\lambda}{\lambda\lambda-16}+\frac{2\lambda}{\lambda\lambda-36}+\text{etc.}$$

hincque

$$\frac{1}{2\lambda\lambda}-\frac{\pi\cot.\frac{1}{2}\lambda\pi}{4\lambda}=\frac{1}{4-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\frac{1}{36-\lambda\lambda}+\text{etc.}$$

Quod si hic loco λ scribamus 2λ, habebimus

$$\frac{1}{8\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{8\lambda}=\frac{1}{4-4\lambda\lambda}+\frac{1}{16-4\lambda\lambda}+\frac{1}{36-4\lambda\lambda}+\text{etc.}$$

siue

$$\frac{1}{2\lambda\lambda}-\frac{\pi\cot.\lambda\pi}{2\lambda}=\frac{1}{1-\lambda\lambda}+\frac{1}{4-\lambda\lambda}+\frac{1}{9-\lambda\lambda}+\frac{1}{16-\lambda\lambda}+\text{etc.}$$

quae series est plane eadem, quam supra § 19 inuenimus.

§. 28. Ponamus nunc vt supra $\zeta=\alpha\pi$ et $\Phi=\lambda\pi$, vt obtineatur sequens series:

$$\frac{\pi\sin.\lambda\pi}{2(\cos.\alpha\pi-\cos.\lambda\pi)}=\frac{\lambda}{(\lambda\lambda-\alpha\alpha)}+\frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha}+\frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha}+\frac{\lambda-4}{(\lambda-4)^2-\alpha\alpha}+\text{etc.}$$

Sin autem hic ponatur $\alpha=\beta\sqrt{-1}$, ista series sequentem induet formam:

$$\frac{\pi\sin.\lambda\pi}{e^{\beta\pi}+e^{-\beta\pi}-2\cos.\lambda\pi}=\frac{\lambda}{\lambda\lambda+\beta\beta}+\frac{\lambda-2}{(\lambda-2)^2+\beta\beta}$$
$$+\frac{\lambda+2}{(\lambda+2)^2+\beta\beta}+\frac{\lambda-4}{(\lambda-4)^2+\beta\beta}+\text{etc.}$$

§. 29.

§. 29. Quod si igitur proposita fuerit haec fractio: $\frac{\sin.\Phi}{a-\cos.\Phi}$ siue $\frac{\sin.\lambda\pi}{a+\cos.\lambda\pi}$, iterum duos casus euolui conuenit, alterum quo $a < 1$, alterum quo $a > 1$. Priore quidem casu, quo $a < 1$, statuatur $\cos.\alpha\pi = a$, vnde fit $\alpha = \frac{\text{Arc.co.}a}{\pi}$, quo inuento erit

$$\frac{\sin.\lambda\pi}{a-\cos.\lambda\pi} = \frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda-\alpha\alpha} + \frac{\lambda-2}{(\lambda-2)^2-\alpha\alpha} + \frac{\lambda+2}{(\lambda+2)^2-\alpha\alpha} + \text{etc.}\right)$$

Sin autem $a > 1$, quaeri debet β, ita vt sit vt ante

$$\beta = \frac{1}{\pi} l(a + \surd(aa-1))$$

quo valore inuento erit

$$\frac{\sin\lambda\pi}{a-\cos.\lambda\pi} = \frac{2}{\pi}\left(\frac{\lambda}{\lambda\lambda+\beta\beta} + \frac{\lambda-2}{(\lambda-2)^2+\beta\beta} + \frac{\lambda+2}{(\lambda+2)^2+\beta\beta} + \text{etc.}\right)$$

Sin autem fuerit $a = 1$, tum fit tam $\alpha = 0$ quam $\beta = 0$, eademque series resultat, quam supra ex casu $\zeta = 0$ eliciumus. Hinc ergo si sumatur $\lambda = \frac{1}{2}$ prodibit series

$$\frac{\pi}{2\cos.\alpha\pi} = \frac{2}{1-4\alpha\alpha} - \frac{6}{9-4\alpha\alpha} + \frac{10}{25-4\alpha\alpha} - \frac{14}{49-4\alpha\alpha} + \text{etc.}$$

vel etiam haec:

$$\frac{\pi}{e^{\beta\pi}-e^{-\beta\pi}} = \frac{2}{1+4\beta\beta} - \frac{6}{9+4\beta\beta} + \frac{10}{25+4\beta\beta} - \frac{14}{49+4\beta\beta} + \text{etc.}$$

Ceterum per se intelligitur, per differentiationem plurimas alias series formari posse.

III. Sit fractio resoluenda $\frac{1}{\cos.\Phi-\cos.2\Phi}$.

§. 30. Ante omnia igitur hic quaeri debet, quibusnam casibus iste denominator euanescat. Cum igitur in genere sit $\cos.\Phi = \cos.(i\pi \pm \Phi)$, denotante i numerum parem, similique modo $\cos.2\Phi = \cos.(i'\pi \pm 2\Phi)$, habebimus $i\pi \pm \Phi = i'\pi \pm 2\Phi$, vnde ob ambiguitatem signorum sequentes casus eruuntur: $\Phi = i\pi$, $\Phi = \frac{i\pi}{3}$. Hic autem probe be eſt obſeruandum, caſus priores bis occurrere, ſeu factores hinc natos $\varphi - i\pi$ bis eſſe collocandos, ita vt factor denominatoris ſit $(\varphi - i\pi)^2$. Quod cum minus clare appareat, ita oſtendamus: quoniam in genere eſt

$$\cos a - \cos b = 2 \sin \tfrac{a+b}{2}, \sin \tfrac{b-a}{2}$$

erit noſter denominator $2 \sin \frac{1}{2}\Phi \sin \frac{3}{2}\varphi$, qui igitur euaneſcit tam quando $\sin \frac{1}{2}\Phi = 0$ quam quando $\sin \frac{3}{2}\varphi = 0$. Fit autem $\sin \frac{\Phi}{2} = 0$ quoties $\frac{\Phi}{2} = i\pi$, denotante i omnes numeros integros, ideoque $\varphi = 2i\pi$. Similique modo $\sin \frac{3\Phi}{2}$ euaneſcit, ſi $\frac{3\Phi}{2} = i\pi$ ideoque $\varphi = \frac{2i\pi}{3}$, quae poſterior formula, quoties $\frac{i}{3}$ eſt numerus integer, priores caſus ſuppeditat; ſicque manifeſtum eſt, in factoribus occurrere omnia quadrata $(\varphi - i\pi)^2$. Reliqui vero factores $\varphi - \frac{2i\pi}{3}$, quando i per 3 non eſt diuiſibile, erunt ſimplices.

§. 31. Cum igitur formula $(\varphi - 2i\pi)^2$ ſit factor noſtri denominatoris $\cos \varphi - \cos 2\varphi$, ſecundum regulam pro huiusmodi caſibus ſtatuamus

$$\frac{1}{\cos \Phi - \cos 2\Phi} = \frac{\alpha}{(\Phi - 2i\pi)^2} + \frac{\beta}{\Phi - 2i\pi} + \mathrm{R},$$

vbi R complectitur omnes reliquas fractiones. Nunc vtrinque multiplicemus per $(\varphi - 2i\pi)^2$ et habebimus

$$\frac{(\Phi - 2i\pi)^2}{\cos \Phi - \cos 2\Phi} = \alpha + \beta(\varphi - 2i\pi) + \mathrm{R}(\varphi - 2i\pi)^2$$

Faciamus $\varphi = 2i\pi$ fietque $\alpha = \frac{(\Phi - 2i\pi)^2}{\cos \Phi - \cos 2\Phi}$, cuius fractionis numerator et denominator euaneſcent, hinc differentialibus ſubſtitutis fiet $\alpha = \frac{2(\Phi - 2i\pi)}{-\sin \Phi + 2\sin 2\Phi}$, vbi cum numerator et denominator iterum euaneſcant, denuo eorum loco differentialia ſcribantur eritque $\alpha = \frac{2}{-\cos \Phi + 4\cos 2\Phi}$. Nunc igitur poſito $\varphi = 2i\pi$ reperietur $\alpha = \frac{2}{3}$.

§. 32.

§. 32. Iam in aequatione

$$\frac{(\Phi-2i\pi)^2}{\cos.\Phi-\cos.2\Phi}=\alpha+\beta(\Phi-2i\pi)+R(\Phi-2i\pi)^2$$

terminus $\alpha=\frac{2}{3}$ ad alteram partem transferatur et ad eandem denominationem reducatur et resultabit haec aequatio:

$$\frac{(\Phi-2i\pi)^2-\frac{2}{3}(\cos.\Phi-\cos.2\Phi)}{\cos.\Phi-\cos.2\Phi}=\beta(\Phi-2i\pi)+R(\Phi-2i\pi)^2$$

vnde per $\Phi-2i\pi$ diuidendo fiet

$$\frac{(\Phi-2i\pi)^2-\frac{2}{3}(\cos.\Phi-\cos.2\Phi)}{(\Phi-2i\pi)(\cos.\Phi-\cos.2\Phi)}=\beta+R(\Phi-2i\pi).$$

Quod si iam statuatur $\Phi=2i\pi$, β aequabitur fractioni, cuius tam numerator quam denominator ter euanescit, ita vt triplici differentiatione sit opus.

Prima autem differentiatio dabit:

$$\beta=\frac{2(\Phi-2i\pi)+\frac{2}{3}(\sin.\Phi-2\sin.2\Phi)}{\cos.\Phi-\cos.2\Phi-(\Phi-2i\pi)(\sin.\Phi-2\sin.2\Phi)}.$$

Secunda differentiatio dabit:

$$\beta=\frac{2+\frac{2}{3}(\cos.\Phi-4\cos.2\Phi)}{-2\sin.\Phi+4\sin.2\Phi-(\Phi-2i\pi)(\cos.\Phi-4\cos.2\Phi)}.$$

Tertia denique differentiatio dat:

$$\beta=\frac{-\frac{2}{3}(\sin.\Phi-8\sin.2\Phi)}{-3\cos.\Phi+12\cos.2\Phi+(\Phi-2i\pi)(\sin.\Phi-8\sin.2\Phi)}.$$

Nunc autem facto $\Phi=2i\pi$ numerator quidem iterum euanescit, denominator vero euadit 9, ita vt sit $\beta=0$.

§. 33. At vero iste valor pro β sine differentiatione facilius erui potest ponendo $\Phi=2i\pi+\omega$, existente ω infinite paruo; tum autem erit

$$\text{cof.}\,\Phi = \text{cof.}\,\omega \text{ et } \text{cof.}\,2\,\Phi = \text{cof.}\,2\,\omega;$$

aequatio autem fiet

$$\frac{\omega\omega}{\text{cof.}\,\omega - \text{cof.}\,2\,\omega} = \tfrac{2}{3} + \beta\,\omega + \text{R}\,\omega\,\omega.$$

Nunc ambos cofinus proxime exhibeamus vsque ad quartam poteftatem ipfius ω procedendo, et cum fit

$$\text{cof.}\,\omega = 1 - \tfrac{1}{2}\,\omega\,\omega + \tfrac{1}{24}\,\omega^4 \text{ et}$$
$$\text{cof.}\,2\,\omega = 1 - 2\,\omega\,\omega + \tfrac{16}{24}\,\omega^4, \text{ erit}$$
$$\text{cof.}\,\omega - \text{cof.}\,2\,\omega = \tfrac{3}{2}\,\omega\,\omega - \tfrac{5}{8}\,\omega^4 = \tfrac{3}{2}\,\omega\,\omega(1 - \tfrac{5}{12}\,\omega\,\omega),$$

quo valore fubftituto habebimus

$$\frac{2}{3(1 - \frac{5}{12}\,\omega\,\omega)} = \tfrac{2}{3}(1 + \tfrac{5}{12}\,\omega\,\omega) = \tfrac{2}{3} + \beta\,\omega + \text{R}\,\omega\,\omega,$$

hincque fit $\beta = \frac{5}{18}\,\omega$; ficque facto $\omega = 0$ erit etiam $\beta = 0$.

§. 34. Hanc obrem pro denominatoris factore quadrato $(\Phi - 2\,i\,\pi)^2$ ob $\alpha = \frac{2}{3}$ fractio inde nata erit $\frac{2}{3(\Phi - 2\,i\,\pi)^2}$. Pro reliquis autem factoribus fimplicibus $\Phi - \frac{2}{3}\,i\,\pi$ ftatuamus

$$\frac{1}{\text{cof.}\,\Phi - \text{cof.}\,2\,\Phi} = \frac{\alpha}{\Phi - \frac{2}{3}\,i\,\pi} + \text{R},$$

quae aequatio multiplicetur per $\Phi - \frac{2}{3}\,i\,\pi = \omega$, vt prodeat

$$\frac{\omega}{\text{cof.}\,\Phi - \text{cof.}\,2\,\Phi} = \alpha + \text{R}\,\omega.$$

Vbi notetur numerum i non effe per 3 diuifibilem, vnde $\frac{2\,i\,\pi}{3}$ fequentes angulos exprimet:

$$\tfrac{2}{3}\pi,\ \tfrac{4}{3}\pi,\ \tfrac{8}{3}\pi,\ \tfrac{10}{3}\pi,\ \tfrac{11}{3}\pi,$$

at anguli $\frac{4\,i\,\pi}{3}$ valores funt $\frac{4\,\pi}{3}$, $\frac{8\,\pi}{3}$, $\frac{16\,\pi}{3}$ quorum angulorum cofinus eft idem $-\frac{1}{2}$, finus autem horum angulorum funt fin. $\frac{2\,i\,\pi}{3} = \pm\frac{\sqrt{3}}{2}$, vbi fignum fuperius valet, fi i fit $3\,n + 1$, inferius vero fi fuerit $i = 3\,n + 2$. At vero fin. $\frac{4\,i\,\pi}{3}$ femper eft $\mp\frac{\sqrt{3}}{2}$, vbi iterum fignum fuperius valet

fi

si $i = 3n + 1$, inferius vero si $i = 3n + 2$. Haecque regula semper valet, siue n sit numerus positiuus siue negatiuus.

§. 35. His praenotatis erit

$$\cos.\,\phi = -\tfrac{1}{2}\cos.\,\omega \mp \tfrac{\surd 3}{2}\sin.\,\omega \text{ et}$$

$$\cos.\,2\phi = -\tfrac{1}{2}\cos.\,2\omega \pm \tfrac{\surd 3}{2}\sin.\,2\omega,$$

vnde vero proxime habebimus

$$\cos.\,\phi = -\tfrac{1}{2}(1 - \tfrac{1}{2}\omega\omega) \mp \tfrac{\surd 3}{2}\omega \text{ et}$$

$$\cos.\,2\phi = -\tfrac{1}{2}(1 - 2\omega\omega) \pm \tfrac{\surd 3}{2}.\,2\omega,$$

vbi perpetuo signa superiora valent si $i = 3n + 1$, inferiora autem si $i = 3n + 2$. Hinc igitur erit noster denominator

$$\cos.\,\phi - \cos.\,2\phi = -\tfrac{3}{4}\omega\omega \mp \tfrac{3\surd 3}{2}.\,\omega$$

vnde fit $\frac{1}{-\frac{3}{4}\omega \mp \frac{3\surd 3}{2}} = \alpha$. Posito igitur $\omega = 0$ erit $\alpha = \mp \frac{2}{3\surd 3}$, ita vt ex factore $\phi - \frac{2i\pi}{3}$ nascatur ista fractio:

$$\mp \frac{2}{3\surd 3}\left(\frac{1}{\phi - \frac{2i\pi}{3}}\right) = \mp \frac{2}{(3\phi - 2i\pi)\surd 3}.$$

§. 36. Euoluamus igitur primo omnes terminos seriei ex factoribus geminatis $(\phi - 2i\pi)^2$ natos, et cum numerator fuisset $\frac{2}{3}$, si loco i successiue omnes scribamus numeros integros tam positiuos quam negatiuos, series orietur sequens:

$$\frac{2}{3\phi\phi} + \frac{2}{3(\phi - 2\pi)^2} + \frac{2}{3(\phi + 2\pi)^2} + \frac{2}{3(\phi - 4\pi)^2} + \frac{2}{3(\phi + 4\pi)^2} + \frac{2}{3(\phi - 6\pi)^2} + \text{etc.}$$

Pro

Pro altera serie sit 1°) $i = 3n + 1$, hincque fractio fiet $-\frac{2}{(3\Phi - 2(3n+1)\pi)\sqrt{3}}$; sin autem sit $i = -3n - 1$, signum inferius valebit et fractio erit $+\frac{2}{(3\Phi + 2(3n+1)\pi)\sqrt{3}}$, qui duo termini contracti praebent $-\frac{8(3n+1)\pi}{(9\Phi\Phi - 4(3n+1)^2\pi\pi)\sqrt{3}}$; sin autem fuerit $i = 3n + 2$, tum vero etiam $i = -3n - 2$, binae fractiones in vnam contractae praebent

$$+\frac{8(3n+2)\pi}{(9\Phi\Phi - 4(3n+2)^2\pi\pi)\sqrt{3}}.$$

Quare cum valores negatiuos ipsius i iam simus complexi, loco n tantum omnes numeros positiuos 0, 1, 2, 3, 4, 5 etc. poni oportet, vnde sequens resultabit series:

$$-\frac{8\pi}{(9\Phi\Phi - 4\pi\pi)\sqrt{3}} - \frac{8\cdot 4\pi}{(9\Phi\Phi - 4\cdot 16\pi\pi)\sqrt{3}} - \frac{8\cdot 7\pi}{(9\Phi\Phi - 4\cdot 49\pi\pi)\sqrt{3}}$$
$$+\frac{16\cdot\pi}{(9\Phi\Phi - 16\pi\pi)\sqrt{3}} + \frac{8\cdot 5\pi}{(9\Phi\Phi - 4\cdot 25\pi\pi)\sqrt{3}} + \frac{8\cdot 8\pi}{(9\Phi\Phi - 4\cdot 64\pi\pi)\sqrt{3}}.$$

§. 37. Proposita igitur fractio $\frac{1}{\cos.\Phi - \cos.2\Phi}$ resoluitur in has duas series

$$\tfrac{2}{3}\left(\frac{1}{\Phi^2} + \frac{1}{(\Phi - 2\pi)^2} + \frac{1}{(\Phi + 2\pi)^2} + \frac{1}{(\Phi - 4\pi)^2} + \frac{1}{(\Phi + 4\pi)^2} + \text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\Phi\Phi - 4\cdot 1\pi\pi} + \frac{4}{9\Phi\Phi - 4\cdot 4^2\pi\pi} + \frac{7}{9\Phi\Phi - 4\cdot 7^2\pi\pi} + \frac{10}{9\Phi\Phi - 4\cdot 10^2\pi\pi} + \text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\Phi\Phi - 4\cdot 2^2\pi\pi} + \frac{5}{9\Phi\Phi - 4\cdot 5^2\pi\pi} + \frac{8}{9\Phi\Phi - 4\cdot 8^2\pi\pi} + \frac{11}{9\Phi\Phi - 4\cdot 11^2\pi\pi} + \text{etc.}\right)$$

Hinc ergo si faciamus vt supra $\Phi = \lambda\pi$ erit fractio

$$\frac{\pi\pi}{\cos.\lambda\pi - \cos.2\lambda\pi} =$$
$$\tfrac{2}{3}\left(\frac{1}{\lambda^2} + \frac{1}{(\lambda-2)^2} + \frac{1}{(\lambda+2)^2} + \frac{1}{(\lambda-4)^2} + \frac{1}{(\lambda+4)^2} + \frac{1}{(\lambda-6)^2} + \frac{1}{(\lambda+6)^2} + \text{etc.}\right)$$
$$-\frac{8\pi}{\sqrt{3}}\left(\frac{1}{9\lambda\lambda - 4\cdot 1^2} + \frac{4}{9\lambda\lambda - 4\cdot 4^2} + \frac{7}{9\lambda\lambda - 4\cdot 7^2} + \frac{10}{9\lambda\lambda - 4\cdot 10^2} + \text{etc.}\right)$$
$$+\frac{8\pi}{\sqrt{3}}\left(\frac{2}{9\lambda\lambda - 4\cdot 2^2} + \frac{5}{9\lambda\lambda - 4\cdot 5^2} + \frac{8}{9\lambda\lambda - 4\cdot 8^2} + \frac{11}{9\lambda\lambda - 4\cdot 11^2} + \text{etc.}\right)$$

§. 38. Vt exemplum afferamus sit $\lambda = \frac{1}{3}$, vt fiat $9\lambda\lambda = 1$ ac prodibit ista summatio:

$\pi\pi$

$$\pi\pi = \frac{2}{3}\left(\frac{9}{1^2} + \frac{9}{5^2} + \frac{9}{7^2} + \frac{9}{11^2} + \frac{9}{13^2} + \frac{9}{17^2} + \text{etc.}\right)$$
$$+ \frac{8\pi}{\sqrt{3}}\left(\frac{1}{4.\,1^2-1} + \frac{4}{4.\,4^2-1} + \frac{7}{4.\,7^2-1} + \frac{10}{4.\,10^2-1} + \frac{13}{4.\,13^2-1} + \text{etc.}\right)$$
$$- \frac{8\pi}{\sqrt{3}}\left(\frac{2}{4.\,2^2-1} + \frac{5}{4.\,5^2-1} + \frac{8}{4.\,8^2-1} + \frac{11}{4.\,11^2-1} + \frac{14}{4.\,14^2-1} + \text{etc.}\right)$$

quae summatio etiam hoc modo referri potest:

$$\pi\pi = 6\left(\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.}\right)$$
$$+ \frac{4\pi}{\sqrt{3}}\left(\frac{2}{2^2-1} - \frac{4}{4^2-1} + \frac{8}{8^2-1} - \frac{10}{10^2-1} + \frac{14}{14^2-1} - \frac{16}{16^2-1} + \text{etc.}\right)$$

elegantior autem forma erit sequens:

$$\pi\pi = 6\left(\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.}\right)$$
$$+ \frac{2\pi}{\sqrt{3}}\left(1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \text{etc.}\right)$$

§. 39. Quoniam hoc casu occurrerunt factores quadrati, etiam eiusmodi fractiones resoluere poterimus, quarum denominatores ipsi sunt quadrati, ideoque meros factores simplices quadratos inuoluunt. Atque adeo hanc resolutionem extendere licebit ad denominatores cubicos altiorumque potestatum, si modo in subsidium vocentur ea praecepta, quae pro huiusmodi resolutionibus olim dedi.

IV. Sit fractio resoluenda proposita $\frac{1}{\text{fin.}\,\Phi^2}$.

§. 40. Cum igitur hic omnes factores quadrati denominatoris in hac forma contineantur: $\frac{1}{(\Phi - i\pi)^2}$, denotante i omnes numeros integros tam positiuos quam negatiuos, ponamus pro resolutione generali

$$\frac{1}{\text{fin.}\,\Phi^2} = \frac{\alpha}{(\Phi - i\pi)^2} + \frac{\beta}{\Phi - i\pi} + \text{R}$$

vbi R complectitur omnes fractiones reliquas. Hinc per $(\Phi - i\pi)^2$ multiplicando erit

$$\frac{(\Phi - i\pi)^2}{\text{fin.}\,\Phi^2} = \alpha + \beta(\Phi - i\pi) + \text{R}(\Phi - i\pi)^2.$$

 Iam

Iam fiat $\phi = i\pi$, et quoniam hoc casu numerator ac denominator nostrae fractionis euanescunt, statuamus $\Phi - i\pi = \omega$, eritque

$$\text{sin.}\,\Phi = \text{sin.}\,(i\pi + \omega) = \text{sin.}\,i\pi\ \text{cos.}\,\omega + \text{sin.}\,\omega\ \text{cos.}\,i\pi = \pm\ \text{sin.}\,\omega$$

ob $\text{sin.}\,i\pi = 0$ et $\text{cos.}\,i\pi = \pm 1$; vbi signum superius valet si i sit numerus par, inferius vero si impar, quod tamen discrimen hic non in censum venit, cum sit $\text{sin.}\,\Phi^2 = \text{sin.}\,\omega^2$. Hinc igitur erit

$$\frac{\omega\omega}{\text{sin.}\,\omega^2} = \alpha + \beta\omega + R\omega\omega.$$

Cum igitur sit

$$\text{sin.}\,\omega = \omega - \tfrac{1}{6}\omega^3 = \omega(1 - \tfrac{1}{6}\omega\omega),\ \text{erit}$$

$$\frac{1}{(1 - \frac{1}{6}\omega\omega)^2} = 1 + \tfrac{1}{3}\omega\omega = \alpha + \beta\omega + R\omega\omega$$

vnde fit statim $\alpha = 1$. Tum vero aequatio erit $\frac{1}{3}\omega = \beta + R\omega$, sicque facto $\omega = 0$ fit $\beta = 0$, consequenter ex denominatoris factore $(\Phi - i\pi)^2$ oritur haec fractio $\frac{1}{(\Phi - i\pi)^2}$.

§. 41. Tribuantur nunc ipsi i omnes valores debiti ac reperietur haec series:

$$\frac{1}{\text{sin.}\,\Phi^2} = \frac{1}{\Phi\Phi} + \frac{1}{(\Phi - \pi)^2} + \frac{1}{(\Phi + \pi)^2} + \frac{1}{(\Phi - 2\pi)^2} + \frac{1}{(\Phi + 2\pi)^2} + \frac{1}{(\Phi - 3\pi)^2} + \text{etc.}$$

quae quidem series deduci potuisset ex § 18, vbi inuenimus

$$\frac{\text{cos.}\,\Phi}{\text{sin.}\,\Phi} = \frac{1}{\Phi} + \frac{1}{\Phi - \pi} + \frac{1}{\Phi + \pi} + \frac{1}{\Phi - 2\pi} + \frac{1}{\Phi + 2\pi} + \text{etc.},$$

vnde per differentiationem signis mutatis ea ipsa oritur series quam hic inuenimus.

§. 42. Quod si fractio fuisset proposita $\frac{\text{cos.}\,\Phi^2}{\text{sin.}\,\Phi^2}$ et eodem modo resolutio instituatur, ob

$$\cos.(i\pi + \omega) = \pm \cos.\omega \text{ ideoque}$$
$$\cos.\Phi^2 = \cos.\omega^2 = 1 - \omega\omega$$

quoniam secundae potestates ipsius ω non in computum veniunt, numerator foret vt in casu praecedente $= 1$, ideoque eadem plane series prodiisset, id quod vtique foret absurdum. Supra autem iam animaduertimus, huiusmodi resolutiones veritati non esse consentaneas, nisi quantitas variabilis Φ in numeratore pauciores habeat dimensiones quam in denominatore, quia alioquin praeter seriem fractionum partes integrae essent accessurae, id quod hoc casu manifesto euenit, cum sit $\frac{\cos.\Phi^2}{\sin.\Phi^2} = \frac{1}{\sin\Phi^2} - 1$, ita vt pars integra hoc casu sit $= -1$.

V. Sit fractio resoluenda $= \frac{1}{\sin.\Phi^3}$.

§. 43. Pro hoc ergo casu poni oportebit

$$\frac{1}{\sin.\Phi^3} = \frac{\alpha}{(\Phi - i\pi)^3} + \frac{\beta}{(\Phi - i\pi)^2} + \frac{\gamma}{\Phi - i\pi} + R.$$

Ponamus nunc iterum $\Phi = i\pi + \omega$, et cum sit

$$\frac{1}{\sin.\Phi^3} = \pm \frac{1}{\omega^3}(1 + \tfrac{1}{2}\omega\omega)$$

(vbi ratio signorum legem supra datam seruet) haec resultat aequatio, postquam per ω^3 fuerit multiplicata:

$$\pm \frac{\omega^3(1 + \frac{1}{2}\omega\omega)}{\omega^3} = \alpha + \beta\omega + \gamma\omega\omega + R\omega^3 = 1 + \tfrac{1}{2}\omega\omega;$$

vnde manifesto fit $\alpha = \pm 1$, tum vero $\beta + \gamma\omega + R\omega\omega = \frac{1}{2}\omega$ sicque erit $\beta = 0$ et $\gamma = \pm \frac{1}{2}$. Hoc igitur modo ex denominatoris factore cubico $(\Phi - i\pi)^3$ nascentur hae duae fractiones: $\frac{1}{(\Phi - i\pi)^3} \pm \frac{1}{2(\Phi - i\pi)}$.

§. 44. Tribuamus igitur litterae i successiue omnes valores tam positiuos quam negatiuos atque obtinebimus sequentem resolutionem:

$$\frac{1}{\sin.\Phi^3} = \frac{1}{\Phi^3} - \frac{1}{(\Phi-\pi)^3} - \frac{1}{(\Phi+\pi)^3} + \frac{1}{(\Phi-2\pi)^3} + \frac{1}{(\Phi+2\pi)^3} - \text{etc.}$$
$$+ \frac{1}{2\Phi} - \frac{1}{2(\Phi-\pi)} - \frac{1}{2(\Phi+\pi)} + \frac{1}{2(\Phi-2\pi)} + \frac{1}{2(\Phi+2\pi)} - \text{etc.}$$

Hic obseruasse iuuabit, inferiorem seriem iam supra in primo exemplo esse inuentam; vnde intelligimus fore summam huius seriei $= \frac{1}{2\,\sin.\Phi}$: quamobrem series superior cuborum sola aequabitur huic formulae: $\frac{1}{\sin.\Phi^3} - \frac{1}{2\,\sin.\Phi}$.

§. 45. Egregie hoc quoque conuenit cum principiis supra stabilitis, ex quibus per differentiationem continuo alias nouas series eruere docuimus. Cum enim sit

$$\frac{1}{\sin.\Phi} = \frac{1}{\Phi} - \frac{1}{\Phi-\pi} - \frac{1}{\Phi+\pi} + \frac{1}{\Phi-2\pi} + \frac{1}{\Phi+2\pi} + \text{etc.}$$

hinc deducitur differentiando

$$-\frac{\cos.\Phi}{\sin.\Phi^2} = -\frac{1}{\Phi^2} + \frac{1}{(\Phi-\pi)^2} + \frac{1}{(\Phi+\pi)^2} - \frac{1}{(\Phi-2\pi)^2} - \frac{1}{(\Phi+2\pi)^2} + \text{etc.}$$

atque hinc denuo differentiando

$$\frac{1}{\sin.\Phi} + \frac{2\cos.\Phi^2}{\sin.\Phi^3} = \frac{2}{\Phi^3} - \frac{2}{(\Phi-\pi)^3} - \frac{2}{(\Phi+\pi)^3} + \frac{2}{(\Phi-2\pi)^3} + \frac{2}{(\Phi+2\pi)^3} + \text{etc.}$$

quae reducitur ad hanc formam: $\frac{2}{\sin.\Phi^3} - \frac{1}{\sin.\Phi}$, id quod egregie conuenit cum valore praecedente.

VI. Sit fractio proposita resoluenda $= \frac{1}{\text{tang.}\,\Phi - \sin.\Phi}$.

§. 46. Denominator iste tang. $\Phi -$ sin. Φ manifesto euanescit casibus quibus $\Phi = i\pi$, denotante i omnes numeros integros tam positiuos quam negatiuos, vnde fractiones simplices, quarum denominatores continent istum factorem $\Phi - i\pi$, euadunt infiniti casu $\Phi = i\pi$, dum reliquae fra-

ctiones

ctiones retinent valorem finitum. Haecque consideratio nobis aperit nouam methodum, omnes fractiones simplices inuestigandi. Pro quouis enim tali factore euanescente quaeratur valor ipsius fractionis propositae, qui cum fiat infinitus, ei aequales esse debebunt ii seriei termini, qui eodem casu euadunt infiniti. Hanc ob rem statui oportebit $\Phi - i\pi = \omega$; denotante ω angulum infinite paruum. Quo facto fractio proposita fiet certa quaedam functio ipsius ω, quam secundum eius dimensiones euolui conueniet.

§. 47. Hanc igitur ideam sequentes, duos casus distinguere debemus, prouti i fuerit numerus vel par vel impar, quoniam priore casu fit $\sin. \Phi = \sin. \omega$, posteriore vero casu fit $\sin. \Phi = - \sin. \omega$, dum vtroque casu manet

$$\tang. \Phi = \tang. \omega.$$

Sit igitur primo i numerus impar et erit casu $\Phi = i\pi$ nostra fractio $\frac{1}{\tang. \omega + \sin. \omega}$. Est vero proxime $\tang. \omega = \omega + \frac{1}{3}\omega^3$ et $\sin. \omega = \omega - \frac{1}{6}\omega^3$, vnde ista fractio fiet

$$\frac{1}{2\omega + \frac{1}{6}\omega^3} = \frac{1}{2\omega(1 + \frac{1}{12}\omega\omega)} = \frac{1}{2\omega}(1 - \tfrac{1}{12}\omega\omega).$$

Haec iam expressio sponte praebet has duas fractiones $\frac{1}{2\omega} - \frac{1}{24}\omega$, vnde ob $\omega = \Phi - i\pi$ pro isto factore oritur haec fractio simplex $\frac{1}{2(\Phi - i\pi)}$, quia altera pars euanescit. Quare si nunc loco i ordine scribamus numeros impares, sequentem fractionum seriem adipiscemur:

$$\frac{1}{2(\Phi - \pi)} + \frac{1}{2(\Phi + \pi)} + \frac{1}{2(\Phi - 3\pi)} + \frac{1}{2(\Phi + 3\pi)} + \frac{1}{2(\Phi - 5\pi)} + \text{etc.}$$

§. 48. Sit nunc etiam i numerus par, vnde fit $\tang. \Phi = \tang. \omega$ et $\sin. \Phi = \sin. \omega$, hinc fractio nostra erit $\frac{1}{\tang. \omega - \sin. \omega}$, vbi facta euolutione primi termini ω

se tollunt, ita vt in hoc denominatore infima potestas ipsius ω futura sit ω^3. Atque ob hanc caussam approximationem vlterius continuari oportet quam casu praecedente. Hunc in finem loco tang. ω scribamus $\frac{\sin.\,\omega}{\cos.\,\omega}$, vt fractio nostra sit $\frac{\cos.\,\omega}{\sin.\,\omega - \sin.\,\omega \cos.\,\omega}$. Cum iam sit

$$\sin.\,\omega \cos.\,\omega = \tfrac{1}{2} \sin.\, 2\,\omega,$$ erit per series

$$\sin.\,\omega = \omega - \tfrac{1}{6}\,\omega^3 + \tfrac{1}{120}\,\omega^5 \text{ et}$$

$$\sin.\, 2\,\omega = 2\,\omega - \tfrac{8}{6}\,\omega^3 + \tfrac{32}{120}\,\omega^5.$$

vnde totus denominator erit

$$+ \tfrac{1}{2}\,\omega^3 - \tfrac{1}{8}\,\omega^5 = \tfrac{1}{2}\,\omega^3 (1 - \tfrac{1}{4}\,\omega\,\omega)$$

numerator vero est cos. $\omega = 1 - \frac{1}{2}\,\omega\,\omega$, vnde tota fractio nostra erit

$$\frac{1 - \frac{1}{2}\,\omega\,\omega}{\frac{1}{2}\,\omega^3 (1 - \frac{1}{4}\,\omega\,\omega)} = \frac{1 - \frac{1}{4}\,\omega\,\omega}{\frac{1}{2}\,\omega^3};$$

hincque partes resultantes erunt $\frac{2}{\omega^3} - \frac{1}{2\,\omega}$, quae ambae casu $\omega = 0$ fiunt infinitae. Facile autem patet, si approximationem vlterius extendissemus, in sequenti termino litteram ω iam in numeratorem transituram fuisse. Scribatur igitur $\Phi - i\pi$ loco ω, et partes ex hoc factore denominatoris oriundae erunt $\frac{2}{(\Phi - i\pi)^3} - \frac{1}{2(\Phi - i\pi)}$, vnde loco i successiue omnes numeros pares scribendo ista prodibit series geminata:

$$\frac{2}{\Phi^3} + \frac{2}{(\Phi - 2\pi)^3} + \frac{2}{(\Phi + 2\pi)^3} + \frac{2}{(\Phi - 4\pi)^3} + \frac{2}{(\Phi + 4\pi)^3} + \frac{2}{(\Phi - 6\pi)^3} + \text{etc.}$$
$$- \frac{1}{2\Phi} - \frac{1}{2(\Phi - 2\pi)} - \frac{1}{2(\Phi + 2\pi)} - \frac{1}{2(\Phi - 4\pi)} - \frac{1}{2(\Phi + 4\pi)} - \text{etc.}$$

§. 49. Iungamus igitur has series ex vtroque casu deductas et fractio proposita $\frac{1}{\text{tang.}\,\Phi - \sin.\,\Phi}$ resolui reperitur in ternas series sequentes:

$$\frac{1}{2(\Phi-\pi)}+\frac{1}{2(\Phi+\pi)}+\frac{1}{2(\Phi-3\pi)}+\frac{1}{2(\Phi+3\pi)}+\frac{1}{2(\Phi-5\pi)}+\text{etc.}$$
$$-\frac{1}{2\Phi}-\frac{1}{2(\Phi-2\pi)}-\frac{1}{2(\Phi+2\pi)}-\frac{1}{2(\Phi-4\pi)}-\frac{1}{2(\Phi+4\pi)}-\text{etc.}$$
$$+\frac{2}{\Phi^3}+\frac{2}{(\Phi-2\pi)^3}+\frac{2}{(\Phi+2\pi)^3}+\frac{2}{(\Phi-4\pi)^3}+\frac{2}{(\Phi+4\pi)^3}+\text{etc.}$$

§. 50. Qnilibet hic facile fentiet, iſtam methodum non parum anteceſſere illi, qua ante vſi ſumus, quandoquidem hoc modo ſtatim fractiones ex quolibet denominatoris factore oriundas nacti ſumus, neque opus fuerat earum numeratores per litteras indefinitas deſignare. Praeterea etiam hac ratione non opus erat ſollicite inquirere, quoties ſinguli factores ſimplices in denominatore contineantur, ſiquidem noſtra methodus hoc ſponte declarat.

§. 51. In huiusmodi autem ſeriebus generalibus, vbi quorundam terminorum denominatores certo caſu euaneſcunt ideoque hi termini in infinitum excreſcunt, quaeri ſolet, his terminis ſublatis, quanta futura ſit ſumma reliquorum terminorum. Ita pro caſu quo i eſt numerus impar, terminus $\frac{1}{2(\Phi-i\pi)}$ fit infinitus caſu $\Phi=i\pi$. Hoc igitur termino deleto quaeritur, quanta futura ſit ſumma reliquorum terminorum caſu $\Phi=i\pi$. Ad hanc quaeſtionem ſoluendam ponatur $\Phi-i\pi=\omega$, atque ex §. 47 patet fore

$$\frac{1}{2\omega}-\frac{1}{24}\omega=\frac{1}{2(\Phi-i\pi)}+R$$

vbi R complectitur omnes reliquos terminos, quorum ſumma deſideratur caſu $\Phi=i\pi$. Transferatur igitur terminus $\frac{1}{2(\Phi-i\pi)}=\frac{1}{2\omega}$ in alteram partem ac ſtatim elucet fore

$$R=-\tfrac{1}{24}\omega=0 \text{ ob } \omega=0,$$

ita vt omiſſo termino illo infinito ſumma omnium reliquorum caſu $\Phi=i\pi$ ſemper ſit 0.

§. 52.

§. 52. Quando autem i est numerus par, eadem conclusio locum habebit, ad quod ostendendum necesse est approximationem adhibitam vlterius continuare. Tum autem erit numerator

$$\operatorname{cos.}\omega = 1 - \tfrac{1}{2}\omega\omega + \tfrac{1}{24}\omega^4;$$

pro denominatore vero

$$\operatorname{sin.}\omega = \omega - \tfrac{1}{6}\omega^3 + \tfrac{1}{120}\omega^5 - \tfrac{1}{5040}\omega^7 \text{ et}$$

$$\operatorname{sin.} 2\omega = 2\omega - \tfrac{8}{6}\omega^3 + \tfrac{32}{120}\omega^5 - \tfrac{128}{5040}\omega^7,$$

vnde fit ipse denominator

$$\tfrac{1}{2}\omega^3 - \tfrac{1}{8}\omega^5 + \tfrac{1}{80}\omega^7 = \tfrac{1}{2}\omega^3\left(1 - \tfrac{1}{4}\omega\omega + \tfrac{1}{40}\omega^4\right);$$

hinc factor posterior in numeratorem translatus praebet

$$1 + \tfrac{1}{4}\omega\omega + \tfrac{3}{80}\omega^4$$

hincque tota fractio iam erit

$$\frac{1 - \tfrac{1}{4}\omega\omega - \tfrac{11}{240}\omega^4}{\tfrac{1}{2}\omega^3}$$

quae aequari debet toti seriei posito $\Phi = i\pi$, hoc est terminis inuentis $\frac{2}{(\Phi - i\pi)^3} - \frac{1}{2(\Phi - i\pi)}$ cum omnibus reliquis R, vnde elicitur $R = -\frac{11}{120}\omega = 0$; vnde patet etiam his casibus summam omnium reliquorum esse $= 0$.

§. 53. Quod si ergo sumamus $\Phi = 0$ et terminos in infinitum excrescentes deleamus, termini remanentes erunt

$$-\frac{1}{2\pi} + \frac{1}{2\pi} - \frac{1}{6\pi} + \frac{1}{6\pi} - \frac{1}{10\pi} + \frac{1}{10\pi} - \text{etc.}$$

$$+\frac{1}{4\pi} - \frac{1}{4\pi} + \frac{1}{8\pi} - \frac{1}{8\pi} + \frac{1}{12\pi} - \frac{1}{12\pi} + \text{etc.}$$

$$-\frac{2}{8\pi^3} + \frac{2}{8\pi^3} - \frac{2}{64\pi^3} + \frac{2}{64\pi^3} - \frac{2}{216\pi^3} + \frac{2}{216\pi^3} - \text{etc.}$$

vbi omnes termini manifesto se tollunt, id quod etiam omnibus reliquis casibus, quibus ponitur $\Phi = i\pi$, contingit.

§. 54.

§. 54. Sin autem binos terminos contiguos contraxissemus, hae series prodiissent:

$$-\frac{1}{2\Phi}+\frac{\Phi}{\Phi\Phi-\pi\pi}-\frac{\Phi}{\Phi\Phi-4\pi\pi}+\frac{\Phi}{\Phi\Phi-9\pi\pi}-\frac{\Phi}{\Phi\Phi-16\pi\pi}+\frac{\Phi}{\Phi\Phi-25\pi\pi}-\text{etc.}$$

$$+\frac{2}{\Phi^3}+\frac{4\Phi(\Phi\Phi+3.4\pi\pi)}{(\Phi\Phi-4\pi\pi)^3}+\frac{4\Phi(\Phi\Phi+3.16\pi\pi)}{(\Phi\Phi-16\pi\pi)^3}+\frac{4\Phi(\Phi\Phi+3.36\pi\pi)}{(\Phi\Phi-36\pi\pi)^3}+\text{etc.}$$

quarum serierum summa est $\frac{1}{\tang.\Phi-\sin.\Phi}$. Quod si nunc hic ponamus $\Phi=0$, siue $\Phi=\omega$, quoniam omnes termini sunt diuisibiles per $\Phi=0$, eorumque autem summa inuenta est $-\frac{11}{120}\omega$, si vtrinque per ω diuidamus, summa erit $=-\frac{11}{120}$; ipsae autem series euadent

$$-\frac{1}{\pi\pi}+\frac{1}{4\pi\pi}-\frac{1}{9\pi\pi}+\frac{1}{16\pi\pi}-\frac{1}{25\pi\pi}+\text{etc.}$$

$$-\frac{3.4}{(4\pi\pi)^2}-\frac{3.4}{(16\pi\pi)^2}-\frac{3.4}{(36\pi\pi)^2}-\frac{3.4}{(64\pi\pi)^2}-\text{etc.}$$

§. 55. Mutatis igitur signis et reductis terminis ad formam simplicissimam impetrabimus hanc summationem:

$$\frac{11}{120}=\frac{1}{\pi\pi}\left(1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\text{etc.}\right)$$

$$+\frac{3}{4\pi^4}\left(1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\frac{1}{36^2}+\text{etc.}\right)$$

Notum autem est esse

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}\ \text{etc.}=\frac{\pi\pi}{12}\ \text{et}$$

$$1+\frac{1}{4^2}+\frac{1}{9^2}+\frac{1}{16^2}+\frac{1}{25^2}+\frac{1}{36^2}=\frac{\pi^4}{90}$$

vnde haec aequalitas manifesto in oculos incurrit.

DE TRANSFORMATIONE SERIERVM IN FRACTIONES CONTINVAS;

VBI SIMVL HAEC THEORIA NON MEDIOCRITER AMPLIFICATVR.

§. 1.

Consideremus fractionem continuam quamcunque, quae sit

$$s = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

ac primo quaeramus fractiones simplices, quae continuo propius ad valorem ipsius s accedant, quas ita formemus vt sit

$$\frac{\mathfrak{A}}{\mathfrak{A}} = a;\quad \frac{\mathfrak{B}}{\mathfrak{B}} = a + \frac{1}{b};\quad \frac{\mathfrak{C}}{\mathfrak{C}} = a + \cfrac{1}{b + \cfrac{1}{c}};$$

$$\frac{\mathfrak{D}}{\mathfrak{D}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

Harum igitur fractionum vltima verum valorem fractionis con-

continuae propositae exprimet. Hinc igitur statim patet fore

$$\frac{A}{\mathfrak{A}} = \frac{a}{1}, \quad \frac{B}{\mathfrak{B}} = \frac{ab+1}{b}, \quad \frac{C}{\mathfrak{C}} = \frac{abc+a+c}{bc+1},$$

Quemadmodum autem hae fractiones vlterius progrediantur sequenti modo inquiramus.

§. 2. Euidens hic est, ex fractione prima secundam oriri, si loco a scribatur $a + \frac{1}{b}$; similique modo ex secunda oriri tertiam, si loco b scribatur $b + \frac{1}{c}$; ex tertia vero quartam, si loco c scribatur $c + \frac{1}{d}$, et ita porro. Hinc ergo, si indefinite fractio $\frac{P}{\mathfrak{P}}$ formata sit ex indicibus $a, b, c, d \ldots p$ binaeque sequentes ponantur $\frac{Q}{\mathfrak{Q}}$ et $\frac{R}{\mathfrak{R}}$, quae respondeant indicibus $a, b, c, d, \ldots q$ et $a, b, c, d, e, \ldots r$, manifestum est, ex fractione $\frac{P}{\mathfrak{P}}$ reperiri sequentem $\frac{Q}{\mathfrak{Q}}$, si loco p scribatur $p + \frac{1}{q}$; ex hac vero $\frac{Q}{\mathfrak{Q}}$ oriri sequentem $\frac{R}{\mathfrak{R}}$, si loco q scribatur $q + \frac{1}{r}$. Nunc vero facile patet, in fractione $\frac{P}{\mathfrak{P}}$ tam numeratorem P quam denominatorem $\mathfrak{P}$ omnes litteras $a, b, c, d, \ldots p$ ita inuoluere, vt nulla earum vltra primam dimensionem exsurgat. Si enim omnes indices $a, b, c, d, e,$ vt inaequales spectentur, nullius quadratum vel altior potestas vsquam occurrere poterit.

§. 3. Quamobrem tam in P quam in $\mathfrak{P}$ duplicis generis occurrent termini, dum alii indicem p plane non continent, alii vero eum tanquam factorem inuoluunt; vnde

 nume-

numerator P huiusmodi habebit formam: $M + Np$, similique modo denominator $\mathfrak{P}$ hanc: $\mathfrak{M} + \mathfrak{N}p$, ita vt sit $\frac{P}{\mathfrak{P}} = \frac{M + Np}{\mathfrak{M} + \mathfrak{N}p}$. In hac igitur forma loco p scribamus $p + \frac{1}{q}$ vt obtineamus fractionem $\frac{Q}{\mathfrak{Q}}$, quae ergo, postquam supra et infra per q multiplicauerimus, erit

$$\frac{Q}{\mathfrak{Q}} = \frac{MQ + Npq + N}{\mathfrak{M}q + \mathfrak{N}pq + \mathfrak{N}} = \frac{N + (M + Np)q}{\mathfrak{N} + (\mathfrak{M} + \mathfrak{N}p)q}.$$

Nunc vt hinc sequentem fractionem $\frac{R}{\mathfrak{R}}$ obtineamus, loco q scribamus $q + \frac{1}{r}$, et postquam supra et infra per r multiplicauerimus orietur

$$\frac{R}{\mathfrak{R}} = \frac{Nr + (M + Np)qr + M + Np}{\mathfrak{N}r + (\mathfrak{M} + \mathfrak{N}p)qr + \mathfrak{M} + \mathfrak{N}p}, \text{ siue}$$

$$\frac{R}{\mathfrak{R}} = \frac{M + Np + (N + Mq + Npq)r}{\mathfrak{M} + \mathfrak{N}p + (\mathfrak{N} + \mathfrak{M}q + \mathfrak{N}pq)r}.$$

Cum igitur sit $P = M + Np$, $Q = N + (M + Np)q$, erit $R = P + Qr$. Simili modo cum sit $\mathfrak{P} = \mathfrak{M} + \mathfrak{N}p$ et $\mathfrak{Q} = \mathfrak{N} = (\mathfrak{M} + \mathfrak{N}p)q$ erit $\mathfrak{R} = \mathfrak{P} + \mathfrak{Q}r$. Sicque patet, quomodo quaelibet nostrarum simplicium fractionum ex binis praecedentibus facile formari possit.

§. 4. Ecce igitur demonstrationem satis planam et dilucidam regulae notissimae pro conuersione fractionis continuae in fractiones simplices, vbi tam numeratores quam denominatores secundum eandem legem ex binis praecedentibus formantur. Cum igitur pro ambabus primis fractionibus sit $A = a$, $\mathfrak{A} = 1$, tum vero $B = ab + 1$ et $\mathfrak{B} = b$, ex his duabus fractionibus sequentes omnes facili negotio formari poterunt. Quod quo clarius appareat singulis indicibus a, b, c, d, e etc. fractiones respondentes ordine subscribamus

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{A}{\mathfrak{A}} & \frac{B}{\mathfrak{B}} & \frac{C}{\mathfrak{C}} & \frac{D}{\mathfrak{D}} & \frac{E}{\mathfrak{E}} & \frac{F}{\mathfrak{F}} & \frac{G}{\mathfrak{G}} \end{array}$$

ac tam numeratores quam denominatores secundum eandem legem ex binis praecedentibus sequenti modo determinabuntur

Pro numeratoribus	Pro denominatoribus
$A = b$	$\mathfrak{A} = 1$
$B = Ab + 1$	$\mathfrak{B} = b$
$C = Bc + A$	$\mathfrak{C} = \mathfrak{B}c + \mathfrak{A}$
$D = Cd + B$	$\mathfrak{D} = \mathfrak{C}d + \mathfrak{B}$
$E = De + C$	$\mathfrak{E} = \mathfrak{D}e + \mathfrak{C}$
$F = Ef + D$	$\mathfrak{F} = \mathfrak{E}f + \mathfrak{D}$
etc.	etc.

Vnde perspicuum est, in serie numeratorum terminum primo anteriorem ex lege progressionis esse debere $= 1$, in serie autem denominatorum terminum primo anteriorem esse debere $= 0$, ita vt fractio primam praecedens sit $\frac{1}{0}$.

§. 5. Quoniam per se satis est perspicuum, has fractiones $\frac{A}{\mathfrak{A}}$, $\frac{B}{\mathfrak{B}}$, $\frac{C}{\mathfrak{C}}$, $\frac{D}{\mathfrak{D}}$, etc. continuo propius ad veritatem accedere, ac tandem verum valorem fractionis continuae exhaurire, necesse est vt differentiae inter harum fractionum binas proximas continuo fiant minores, quamobrem has differentias ordine euoluamus. Primo igitur habebimus

$$II - I = \frac{B\mathfrak{A} - A\mathfrak{B}}{\mathfrak{A}\mathfrak{B}}.$$

Iam hic loco B et $\mathfrak{B}$ valores ex tabula substituantur ac prodibit numerator $A\mathfrak{A}b + \mathfrak{A} - Ab$, quae forma ob $\mathfrak{A} = 1$ abit in 1, ita vt sit $\frac{B}{\mathfrak{B}} - \frac{A}{\mathfrak{A}} = \frac{1}{\mathfrak{A}\mathfrak{B}}$. Porro erit

$$\mathrm{III} - \mathrm{II} = \frac{C\mathfrak{B} - B\mathfrak{C}}{\mathfrak{B}\mathfrak{C}}$$

cuius numerator, si loco C et $\mathfrak{C}$ valores assignati scribantur, praebet

$$\mathfrak{B}(Bc + A) - B(\mathfrak{B}c + \mathfrak{A}) = A\mathfrak{B} - B\mathfrak{A}.$$

Modo autem vidimus esse $B\mathfrak{A} - A\mathfrak{B} = 1$, vnde iste numerator erit $= -1$ ideoque $\frac{C}{\mathfrak{C}} - \frac{B}{\mathfrak{B}} = -\frac{1}{\mathfrak{B}\mathfrak{C}}$. Porro est

$$\mathrm{IV} - \mathrm{III} = \frac{D\mathfrak{C} - C\mathfrak{D}}{\mathfrak{C}\mathfrak{D}}$$

vbi, si loco D et $\mathfrak{D}$ valores assignati scribantur, erit

$$\mathfrak{C}D - C\mathfrak{D} = \mathfrak{C}(Cd + B) - C(\mathfrak{C}d + \mathfrak{B}) = B\mathfrak{C} - C\mathfrak{B}.$$

Modo autem vidimus esse $C\mathfrak{B} - B\mathfrak{C} = -1$, vnde concluditur $\frac{D}{\mathfrak{D}} - \frac{C}{\mathfrak{C}} = +\frac{1}{\mathfrak{C}\mathfrak{D}}$. Simili modo reperietur pro sequentibus

$$\frac{E}{\mathfrak{E}} - \frac{D}{\mathfrak{D}} = -\frac{1}{\mathfrak{D}\mathfrak{E}};\quad \frac{F}{\mathfrak{F}} - \frac{E}{\mathfrak{E}} = +\frac{1}{\mathfrak{E}\mathfrak{F}};\ \text{etc.}$$

§. 6. Hinc igitur singulas nostras fractiones ex sola prima $\frac{A}{\mathfrak{A}} = a$ et fractionibus solas litteras germanicas involuentibus definire poterimus, quandoquidem habebimus

$$\frac{B}{\mathfrak{B}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}}$$

$$\frac{C}{\mathfrak{C}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}}$$

$$\frac{D}{\mathfrak{D}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}}$$

$$\frac{E}{\mathfrak{E}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}} - \frac{1}{\mathfrak{D}\mathfrak{E}}$$

$$\frac{F}{\mathfrak{F}} = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}} - \frac{1}{\mathfrak{D}\mathfrak{E}} + \frac{1}{\mathfrak{E}\mathfrak{F}}$$

etc. etc.

§. 7. Cum igitur harum fractionum vltima, seu infinitesima, verum valorem fractionis continuae propositae, quem designamus littera s, exhibeat, erit

$$s = a + \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}} - \frac{1}{\mathfrak{D}\mathfrak{E}} + \frac{1}{\mathfrak{E}\mathfrak{F}} - \frac{1}{\mathfrak{F}\mathfrak{G}} + \text{etc.}$$

sicque fractionem continuam reduximus ad seriem infinitam fractionum, quarum omnes numeratores sunt alternatim $+1$ et -1, denominatores vero per solas litteras germanicas determinantur, ita vt non opus sit valores litterarum A, B, C, euoluere, sed sufficiat sequentes formulas expediuisse:

$$\mathfrak{A} = 1;\ \mathfrak{B} = b;\ \mathfrak{C} = \mathfrak{B}c + \mathfrak{A};\ \mathfrak{D} = \mathfrak{C}d + \mathfrak{B};\ \mathfrak{E} = \mathfrak{D}e + \mathfrak{C};\ \text{etc.}$$

§. 8. Cum igitur vtraque expressio incipiat a quantitate a, ea prorsus ex calculo egredietur, quoniam litterae germanicae ab ea prorsus non pendent; vnde quae hactenus inuenimus huc redeunt, vt proposita fractione continua

$$s = \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

si ex eius indicibus b, c, d, e, etc. definiantur litterae germanicae, vbi quidem continuo est $\mathfrak{A} = 1$, semper futurum sit

$$s = \frac{1}{\mathfrak{A}\mathfrak{B}} - \frac{1}{\mathfrak{B}\mathfrak{C}} + \frac{1}{\mathfrak{C}\mathfrak{D}} - \frac{1}{\mathfrak{D}\mathfrak{E}} + \text{etc.}$$

quae progressio in infinitum progreditur, si fractio continua in infinitum extendatur, contra vero finito terminorum numero constabit.

§. 9. Cum igitur hoc modo fractionem continuam in seriem ordinariam transformauerimus, haud difficile erit, seriem quamcunque propositam in fractionem continuam conuertere. Proposita igitur sit ista series infinita:

$s =$

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \text{etc.}$$

cuius quidem numeratores omnes ſint vnitates ſigno $+$ et $-$ alternatim affectae, denominatores vero progreſſionem quamcunque conſtituant, quod tamen non obſtat, quo minus omnes plane ſeries in hac forma contineantur, ſiquidem termini ſeriei α, β, γ, δ non ſolum numeri fracti, ſed etiam negatiui euadere poſſunt.

§. 10. Quo igitur fractionem continuam iſti ſeriei aequalem eruamus, primo faciamus $\mathfrak{AB} = \alpha$, $\mathfrak{BC} = \beta$, $\mathfrak{CD} = \gamma$, et ita porro, vnde ob $\mathfrak{A} = 1$ ſequentes nanciſcemur valores:

$$\mathfrak{B} = \alpha; \qquad \mathfrak{C} = \frac{\beta}{\alpha};$$
$$\mathfrak{D} = \frac{\alpha\gamma}{\beta}; \qquad \mathfrak{E} = \frac{\beta\delta}{\alpha\gamma};$$
$$\mathfrak{F} = \frac{\alpha\gamma\varepsilon}{\beta\delta}; \qquad \mathfrak{G} = \frac{\beta\delta\zeta}{\alpha\gamma\varepsilon},$$
$$\mathfrak{H} = \frac{\alpha\gamma\varepsilon\eta}{\beta\delta\zeta}; \qquad \mathfrak{I} = \frac{\beta\delta\zeta\theta}{\alpha\gamma\varepsilon\eta};$$
$$\text{etc.} \qquad \text{etc.}$$

Nunc igitur tantum ſupereſt, vt ex his valoribus litterarum germanicarum ipſos indices b, c, d, e fractionis continuae eliciamus.

§. 11. Ex formulis autem, quibus ſupra litterae germanicae per indices fractionis continuae ſunt determinatae viciſſim ex his litteris ipſos indices b, c, d, e, f etc. definiamus, ac reperiemus

$$b = \mathfrak{B}, \; c = \frac{\mathfrak{C} - \mathfrak{A}}{\mathfrak{B}}, \; d = \frac{\mathfrak{D} - \mathfrak{B}}{\mathfrak{C}}, \; e = \frac{\mathfrak{E} - \mathfrak{C}}{\mathfrak{D}}, \; f = \frac{\mathfrak{F} - \mathfrak{D}}{\mathfrak{E}}, \text{ etc.}$$

Hos igitur valores ordine euoluamus, dum loco litterarum $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. formulas ante inuentas ſubſtituemus.

§. 12. Primo autem erat $\mathfrak{B} = \alpha$, vnde fit $b = \alpha$; deinde eſt

$\mathfrak{C}$ —

$$\mathfrak{C} - \mathfrak{A} = \frac{\beta - \alpha}{\alpha}, \text{ vnde fit } c = \frac{\beta - \alpha}{\alpha\alpha}.$$

Porro erit

$$\mathfrak{D} - \mathfrak{B} = \frac{\alpha(\gamma - \beta)}{\beta}, \text{ vnde fit } d = \frac{\alpha\alpha(\gamma - \beta)}{\beta\beta}.$$

Deinde habebimus

$$\mathfrak{E} - \mathfrak{C} = \frac{\beta(\delta - \gamma)}{\alpha\gamma} \text{ hincque } e = \frac{\beta\beta(\delta - \gamma)}{\alpha\alpha\gamma\gamma}.$$

Simili modo ob

$$\mathfrak{F} - \mathfrak{D} = \frac{\alpha\gamma(\varepsilon - \delta)}{\beta\delta} \text{ erit } f = \frac{\alpha\alpha\gamma\gamma(\varepsilon - \delta)}{\beta\beta\delta\delta}.$$

Eodem modo ob

$$\mathfrak{G} - \mathfrak{E} = \frac{\beta\delta(\zeta - \varepsilon)}{\alpha\gamma\varepsilon} \text{ erit } g = \frac{\beta\beta\delta\delta(\zeta - \varepsilon)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon}$$

etc.

Hac igitur ratione indices fractionis continuae, quam quaerimus, sequenti modo erunt expressi:

$$b = \alpha \qquad c = \frac{\beta - \alpha}{\alpha\alpha}$$

$$d = \frac{\alpha\alpha(\gamma - \beta)}{\beta\beta} \qquad e = \frac{\beta\beta(\delta - \gamma)}{\alpha\alpha\gamma\gamma}$$

$$f = \frac{\alpha\alpha\gamma\gamma(\varepsilon - \delta)}{\beta\beta\delta\delta} \qquad g = \frac{\beta\beta\delta\delta(\zeta - \varepsilon)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon}$$

$$h = \frac{\alpha\alpha\gamma\gamma\varepsilon\varepsilon(\eta - \zeta)}{\beta\beta\delta\delta\zeta\zeta} \qquad i = \frac{\beta\beta\delta\delta\zeta\zeta(\theta - \eta)}{\alpha\alpha\gamma\gamma\varepsilon\varepsilon\eta\eta}$$

etc. etc.

§. 13. Tantum igitur opus est vt isti valores loco indicum b, c, d, e, f etc. in fractione continua

$$s = \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}$$

substituamus; quoniam vero isti valores sunt fracti, quo facilius formam a fractionibus partialibus liberemus, primum ex valoribus inuentis denominatores tollamus eritque

$b = \alpha$

$$b = \alpha \qquad \alpha\alpha c = \beta - \alpha$$
$$\beta\beta d = \alpha\alpha(\gamma - \beta), \qquad \alpha\alpha\gamma\gamma e = \beta\beta(\delta - \gamma)$$
$$\beta\beta\delta\delta f = \alpha\alpha\gamma\gamma(\varepsilon - \delta), \qquad \alpha\alpha\gamma\gamma\varepsilon\varepsilon g = \beta\beta\delta\delta(\zeta - \varepsilon)$$
$$\beta\beta\delta\delta\zeta\zeta h = \alpha\alpha\gamma\gamma\varepsilon\varepsilon(\eta - \zeta), \qquad \alpha\alpha\gamma\gamma\varepsilon\varepsilon\eta\eta i = \beta\beta\delta\delta\zeta\zeta(\theta - \eta)$$
$$\text{etc.} \qquad \text{etc.}$$

§. 14. Nunc ipsam fractionem continuam ita transformemus, vt loco indicum eaedem formulae occurrant, quarum valores hic assignauimus. Secundam scilicet fractionem multiplicemus supra et infra per $\alpha\alpha$, tertiam per $\beta\beta$, quartam per $\alpha\alpha\gamma\gamma$, quintam per $\beta\beta\delta\delta$, sextam per $\alpha\alpha\gamma\gamma\varepsilon\varepsilon$, etc. vt prodeat ista forma:

$$s = \cfrac{1}{b + \cfrac{\alpha\alpha}{\alpha\alpha c + \cfrac{\alpha\alpha\beta\beta}{\beta\beta d + \cfrac{\alpha\alpha\beta\beta\gamma\gamma}{\alpha\alpha\gamma\gamma e + \cfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{\beta\beta\delta\delta f + \text{etc.}}}}}}$$

§. 15. Quod si iam loco horum nouorum indicum $\alpha\alpha c$, $\beta\beta d$, $\alpha\alpha\gamma\gamma e$ etc. valores supra inuentos substituamus, sequens oritur fractio continua:

$$s = \cfrac{1}{\alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\alpha\alpha\beta\beta}{\alpha\alpha(\gamma - \beta) + \cfrac{\alpha\alpha\beta\beta\gamma\gamma}{\beta\beta(\delta - \gamma) + \cfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{\alpha\alpha\gamma\gamma(\varepsilon - \delta) + \text{etc}}}}}}$$

Quod

Quod si hanc formam attentius consideremus, deprehendimus, tertiam fractionem supra et infra deprimi posse per $\alpha\alpha$, tum vero quartam per $\beta\beta$, quintam per $\gamma\gamma$, sextam per $\delta\delta$ etc. quo facto oritur haec fractio continua:

$$s = \cfrac{1}{\alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\beta\beta}{\gamma - \beta + \cfrac{\gamma\gamma}{\delta - \gamma + \cfrac{\delta\delta}{\varepsilon - \delta + \text{etc.}}}}}}$$

Hinc igitur stabiliamus sequens

Theorema I.

§. 16. *Si proposita fuerit talis series infinita:*

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \text{etc.}$$

ex ea semper formari poterit talis fractio continua:

$$\frac{1}{s} = \alpha + \cfrac{\alpha\alpha}{\beta - \alpha + \cfrac{\beta\beta}{\gamma - \beta + \cfrac{\gamma\gamma}{\delta - \gamma + \text{etc.}}}}$$

§. 17. Hanc igitur reductionem per plures ambages ex consideratione fractionis continuae elicuimus, quo quidem proposito nostro satisfecimus, quandoquidem seriem quamcunque in fractionem continuam transformauimus. Verum hic merito desideratur methodus directa, qua immediate ex serie proposita sine illis ambagibus fractio continua illi aequalis deriuari possit. Talem igitur methodum, quippe qua Theoria fractionum continuarum non mediocriter illustrabitur, hic sum expositurus.

Problema.

§. 18. *Propositam seriem infinitam*

$$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \text{etc.}$$

in fractionem continuam transformare.

Solutio.

Cum fit

$s = \frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \text{etc.}$, statuamus

$t = + \frac{1}{\beta} - \frac{1}{\gamma} + \frac{1}{\delta} - \frac{1}{\varepsilon} + \text{etc.}$ et

$u = \frac{1}{\gamma} - \frac{1}{\delta} + \frac{1}{\varepsilon} - \frac{1}{\zeta} + \text{etc.}$

Hinc ergo erit $s = \frac{1}{\alpha} - t = \frac{1 - \alpha t}{\alpha}$, vnde fit $\frac{1}{s} = \frac{\alpha}{1 - \alpha t} = \alpha + \frac{\alpha \alpha t}{1 - \alpha t}$. Est autem $\frac{\alpha \alpha t}{1 - \alpha t} = \cfrac{\alpha \alpha}{-\alpha + \cfrac{1}{t}}$, vnde fit $\frac{1}{s} = \alpha + \cfrac{\alpha \alpha}{-\alpha + \cfrac{1}{t}}$. Simili ergo modo erit etiam $\frac{1}{t} = \beta + \cfrac{\beta \beta}{-\beta + \cfrac{1}{u}}$ et $\frac{1}{u} = \gamma + \cfrac{\gamma \gamma}{-\gamma + \cfrac{1}{v}}$, etc. quibus valoribus substitutis obtinebitur sequens fractio continua:

$$\frac{1}{s} = \alpha + \cfrac{\alpha \alpha}{\beta - \alpha + \cfrac{\beta \beta}{\gamma - \beta + \cfrac{\gamma \gamma}{\delta - \gamma + \text{etc.}}}}$$

quae est ipsa forma in theoremate exhibita.

§. 19.

§. 19. Quod si ergo series proposita sit

$$s = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.} = l\,2,$$ ob

$$\alpha = 1,\ \beta = 2,\ \gamma = 3,\ \delta = 4,\ \text{etc. erit}$$

$$\frac{1}{l\,2} = 1 + \cfrac{1 \cdot 1}{1 + \cfrac{2 \cdot 2}{1 + \cfrac{3 \cdot 3}{1 + \text{etc.}}}}$$

Sin autem assumamus hanc seriem:

$$s = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.} = \tfrac{\pi}{4},$$ ob

$$\alpha = 1,\ \beta = 3,\ \gamma = 5,\ \delta = 7,\ \text{etc. erit}$$

$$\frac{4}{\pi} = 1 + \cfrac{1 \cdot 1}{2 + \cfrac{3 \cdot 3}{2 + \cfrac{5 \cdot 5}{2 + \text{etc.}}}}$$

quae est ipsa fractio continua olim a *Brounchero* prolata.

§. 20. Sumamus $s = \int \frac{x^{m-1}\,dx}{1 + x^n}$, et post integrationem statuamus $x = 1$, quo facto valor ipsius s per sequentem seriem exprimetur:

$$s = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \text{etc.}$$

ita vt sit

$$\alpha = m,\ \beta = m+n,\ \gamma = m+2n,\ \delta = m+3n,\ \text{etc.}$$

hinc ergo sequens fractio continua emerget:

$$\frac{1}{s}=m+\cfrac{mm}{n+\cfrac{(m+n)^2}{n+\cfrac{(m+2n)^2}{n+\text{etc.}}}}$$

quem valorem iam XI Tom. Comentar. Vet. nostrae Academiae dedi.

§. 21. Sin autem proposita sit ista series:

$$s=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}+\frac{1}{\varepsilon}+\text{etc.}$$

cuius omnes termini sint positiui, tantum opus est vt in superiore fractione continua loco litterarum β, δ, ζ, θ, scribatur $-\beta$, $-\delta$, $-\zeta$, etc. tum igitur fiet

$$\frac{1}{s}=\alpha+\cfrac{\alpha\alpha}{-\beta-\alpha+\cfrac{\beta\beta}{\gamma+\beta+\cfrac{\gamma\gamma}{-\delta-\gamma+\cfrac{\delta\delta}{\varepsilon+\delta+\text{etc.}}}}}$$

quae fractio facile transmutatur in hanc formam:

$$\frac{1}{s}=\alpha-\cfrac{\alpha\alpha}{\alpha+\beta-\cfrac{\beta\beta}{\beta+\gamma-\cfrac{\gamma\gamma}{\gamma+\delta-\text{etc.}}}}$$

§. 22. Pluribus autem modis ipsa series proposita transformari potest, vnde continuo aliae atque aliae fractiones continuae eliciuntur. Nonnullas igitur huiusmodi formas hic perpendamus. Sit ergo

$$\alpha=ab,\ \beta=bc,\ \gamma=cd,\ \delta=de\ \text{etc.}$$

vt

vt habeatur iſta ſeries:

$$s = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + \text{etc.}$$

hincque formabitur iſta fractio continua:

$$\frac{1}{s} = ab + \cfrac{aabb}{b(c-a) + \cfrac{bbcc}{c(d-b) + \cfrac{ccdd}{d(e-c) + \text{etc.}}}}$$

quae facile reducitur ad formam ſequentem:

$$\frac{1}{s} = ab + \cfrac{aab}{c-a + \cfrac{bc}{d-b + \cfrac{cd}{e-c + \text{etc.}}}}$$

ſiue

$$\frac{1}{as} = b + \cfrac{ab}{c-a + \cfrac{bc}{d-b + \text{etc.}}}$$

quae forma nobis ſuppeditat ſequens theorema:

Theorema II.

§. 23. *Si propoſita fuerit ſeries huius formae:*

$$s = \frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + \frac{1}{ef} - \text{etc.}$$

ex ea ſequens oritur fractio continua:

$$\frac{1}{as} = b + \cfrac{ab}{c-a + \cfrac{bc}{d-b + \cfrac{cd}{e-c + \cfrac{de}{f-d + \text{etc.}}}}}$$

§. 24.

§. 24. Haec forma, etsi facile ex praecedente deriuatur, ideo est notatu digna, quod fractionem continuam formae maxime diuersae praebeat, vnde operae pretium erit exempla supra allata etiam ad hanc formam accommodare. Cum igitur fuerit

$$l2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc., erit}$$

$$l2 - 1 = -\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.}$$

et his seriebus addendis oritur

$$2l2 - 1 = \tfrac{1}{1.2} - \tfrac{1}{2.3} + \tfrac{1}{3.4} - \tfrac{1}{4.5} + \tfrac{1}{5.6} - \text{etc.}$$

Hic ergo est

$$s = 2l2 - 1 \text{ et } a = 1,\ b = 2,\ c = 3,\ d = 4, \text{ etc.}$$

hinc igitur formabitur ista fractio continua:

$$\frac{1}{2l2-1} = 2 + \cfrac{1.2}{2 + \cfrac{2.3}{2 + \cfrac{3.4}{2 + \text{etc.}}}}$$

§. 25. Simili modo quia est

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc. erit}$$

$$\tfrac{\pi}{4} - 1 = -\tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} - \text{etc.}$$

quarum serierum summa dat

$$\tfrac{\pi}{2} - 1 = \tfrac{2}{1.3} - \tfrac{2}{3.5} + \tfrac{2}{5.7} - \tfrac{2}{7.9} + \text{etc. siue}$$

$$\tfrac{\pi}{4} - \tfrac{1}{2} = \tfrac{1}{1.3} - \tfrac{1}{3.5} + \tfrac{1}{5.7} - \tfrac{1}{7.9} + \text{etc.}$$

Hic igitur erit $s = \frac{\pi}{4} - \frac{1}{2}$; tum vero

$$a = 1,\ b = 3,\ c = 5,\ d = 7 \text{ etc.}$$

quare fractio continua hinc nata erit

$\frac{4}{\pi-2}$

$$\frac{4}{\pi-2}=3+\cfrac{1.3}{4+\cfrac{3.5}{4+\cfrac{5.7}{4+\cfrac{7.9}{4+\text{etc.}}}}}$$

§. 26. Generalius nunc etiam hanc transformationem contemplemur. Denotet igitur Δ valorem formulae integralis $\int\frac{x^{m-1}\,dx}{1+x^n}$, posito post integrationem $x=1$, et cum sit vt supra § 20. vidimus

$$\Delta=\frac{1}{m}-\frac{1}{m+n}+\frac{1}{m+2n}-\text{etc. erit}$$

$$\Delta-\frac{1}{m}=-\frac{1}{m+n}+\frac{1}{m+2n}-\frac{1}{m+3n}+\text{etc.}$$

quibus seriebus additis fit

$$2\Delta-\frac{1}{m}=\frac{n}{m(m+n)}-\frac{n}{(m+n)(m+2n)}+\frac{n}{(m+2n)(m+3n)}-\text{etc.}$$

hinc diuidendo per n erit

$$\frac{2m\Delta-1}{mn}=\frac{1}{m(m+n)}-\frac{1}{(m+n)(m+2n)}+\frac{1}{(m+2n)(m+3n)}-\text{etc.}$$

Hic igitur habemus $s=\frac{2m\Delta-1}{mn}$, tum vero

$$a=m,\ b=m+n,\ c=m+2n,\ d=m+3n\ \text{etc.}$$

quocirca fractio continua hinc formata erit

$$\frac{n}{2m\Delta-1}=m+n+\cfrac{m(m+n)}{2n+\cfrac{(m+n)(m+2n)}{2n+\cfrac{(m+2n)(m+3n)}{2n+\cfrac{(m+3n)(m+4n)}{2n+\text{etc.}}}}}$$

quae forma praecedenti simplicitate nihil cedit.

§. 27. Tribuamus nunc etiam seriei initio assumtae

$$\frac{1}{\alpha} - \frac{1}{\beta} + \frac{1}{\gamma} - \frac{1}{\delta} + \text{ etc.}$$

numeratores quoscunque, sitque

$$s = \frac{a}{\alpha} - \frac{b}{\beta} + \frac{c}{\gamma} - \frac{d}{\delta} + \text{ etc.}$$

atque in Theoremate primo loco litterarum $\alpha, \beta, \gamma, \delta$, etc. scribi oportet $\frac{\alpha}{a}, \frac{\beta}{b}, \frac{\gamma}{c}, \frac{\delta}{d}$, etc. quo facto fractio continua ita se habebit:

$$\frac{1}{s} = \frac{\alpha}{a} + \cfrac{\frac{\alpha\alpha}{aa}}{\frac{\beta}{b} - \frac{\alpha}{a} + \cfrac{\frac{\beta\beta}{bb}}{\frac{\gamma}{c} - \frac{\beta}{b} + \cfrac{\frac{\gamma\gamma}{cc}}{\frac{\delta}{d} - \frac{\gamma}{c} + \text{ etc.}}}}$$

Iam ad fractiones tollendas prima fractio supra et infra multiplicetur per ab, secunda per bc, tertia per cd, et ita porro; tum vero utrinque per a multiplicando obtinebitur

$$\frac{a}{s} = \alpha + \cfrac{\alpha\alpha b}{a\beta - b\alpha + \cfrac{ac\beta\beta}{b\gamma - c\beta + \cfrac{bd\gamma\gamma}{c\delta - d\gamma + \text{ etc.}}}}$$

Hinc igitur formetur sequens

Theorema III.

§. 28. *Si proposita fuerit series infinita huius formae*

$$s = \frac{a}{\alpha} - \frac{b}{\beta} + \frac{c}{\gamma} - \frac{d}{\delta} + \text{ etc.}$$

ex ea formabitur sequens fractio continua:

$$\frac{a}{s} =$$

$$\frac{a}{s} = \alpha + \cfrac{\alpha\alpha b}{a\beta - b\alpha + \cfrac{ac\beta\beta}{b\gamma - c\beta + \cfrac{bd\gamma\gamma}{c\delta - d\gamma + \text{etc.}}}}$$

§. 29. Ad hoc illustrandum proposita sit haec series:

$$\frac{1}{1} - \frac{2}{2} + \frac{3}{3} - \frac{4}{4} + \frac{5}{5} - \text{etc.} = \frac{1}{2},$$

ita vt sit $s = \frac{1}{2}$; fractio ergo continua hinc orta erit

$$2 = 1 + \cfrac{2}{0 + \cfrac{3.4}{0 + \cfrac{8.9}{0 + \cfrac{15.16}{0 + \text{etc.}}}}}$$

quae forma reducitur ad istud productum infinitum:

$$2 = 1 + \frac{2.1^2.2.4.3^2.4.6.5^2.6.8.7^2.\text{etc.}}{1.3.2^2.3.5.4^2.5.7.6^2.7.9.8^2.\text{etc.}}$$

cuius veritas non facile perspicitur, quoniam numeri factorum in numeratore et denominatore non aequales statui possunt, etiamsi ambo sint infiniti. Nullum vero dubium esse potest, quin valor istius producti sit $= 1$.

§. 30. Consideremus nunc istam seriem:

$$s = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \text{etc.}$$

cuius summa est $s = l\,2 - \frac{1}{2}$. Quia igitur est

$$a = 1,\ b = 2,\ c = 3,\ d = 4,\ \text{etc.}$$

$$\alpha = 2,\ \beta = 3,\ \gamma = 4,\ \delta = 5,\ \text{etc.}$$

fractio continua hinc nata erit

$$\frac{1}{l\,2-\frac{1}{2}} = 2 + \cfrac{1.\,2.\,2^2}{-1 + \cfrac{1.\,3.\,3^2}{-1 + \cfrac{2.\,4.\,4^2}{-1 + \cfrac{3.\,5.\,5^2}{-1 + \text{etc.}}}}}$$

§. 31. Quod si autem hanc accipiamus seriem:

$s = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \text{etc.}$

cuius valor est $\frac{1}{2} + l\,2$, habebimus

$a = 2,\ b = 3,\ c = 4,\ d = 5$, etc.

$\alpha = 1,\ \beta = 2,\ \gamma = 3,\ \delta = 4$, etc.

hinc ergo orietur haec fractio continua:

$$\frac{2}{\frac{1}{2}+l\,2} = 1 + \cfrac{1.\,3.\,1^2}{1 + \cfrac{2.\,4.\,2^2}{1 + \cfrac{3.\,5.\,3^2}{1 + \cfrac{4.\,6.\,4^2}{1 + \text{etc.}}}}}$$

siue

$$\frac{4}{2\,l\,2+1} = 1 + \cfrac{1^3.\,3}{1 + \cfrac{2^3.\,4}{1 + \cfrac{3^3.\,5}{1 + \text{etc.}}}}$$

Problema II.

Propositam seriem infinitam

$s =$

$$s = \frac{x}{\alpha} - \frac{xx}{\beta} + \frac{x^3}{\gamma} - \frac{x^4}{\delta} + \text{etc.}$$

in fractionem continuam transformare.

Solutio.

§. 32. Confiderentur fequentes feries ex propofita ferie formatae:

$$t = \frac{x}{\beta} - \frac{xx}{\gamma} + \frac{x^3}{\delta} - \frac{x^4}{\varepsilon} + \text{etc., porro}$$

$$u = \frac{x}{\gamma} - \frac{xx}{\delta} + \frac{x^3}{\varepsilon} - \frac{x^4}{\zeta} + \text{etc.}$$

$$v = \frac{x}{\delta} - \frac{xx}{\varepsilon} + \frac{x^3}{\zeta} - \frac{x^4}{\eta} + \text{etc., eritque}$$

$$s = \frac{x}{\alpha} - tx = \frac{x(1-\alpha t)}{\alpha}, \text{ vnde fit}$$

$$\frac{x}{s} = \frac{\alpha}{1-\alpha t} = \alpha + \frac{\alpha\alpha t}{1-\alpha t} = \alpha + \cfrac{\alpha\alpha}{-\alpha + \cfrac{1}{t}}$$

Hinc ergo erit

$$\frac{x}{s} = \alpha + \cfrac{\alpha\alpha x}{-\alpha x + \cfrac{x}{t}}$$

fimili autem modo erit

$$\frac{x}{t} = \beta + \cfrac{\beta\beta x}{-\beta x + \cfrac{x}{u}}$$

Hi ergo valores fi omnes ordine fubftituantur, orietur ifta fractio continua:

$$\frac{x}{s} = \alpha + \cfrac{\alpha\alpha x}{\beta - \alpha x + \cfrac{\beta\beta x}{\gamma - \beta x + \cfrac{\gamma\gamma x}{\delta - \gamma x + \text{etc.}}}}$$

§. 33.

§. 33. Quod si hic vbique loco x scribamus $\frac{x}{y}$, vt habeamus hanc seriem:

$$s = \frac{x}{\alpha y} - \frac{xx}{\beta yy} + \frac{x^3}{\gamma y^3} - \frac{x^4}{\delta y^4} + \text{etc.}$$

tum fractio continua hinc nata erit

$$\frac{x}{sy} = \alpha + \cfrac{\alpha\alpha x : y}{\beta - \frac{\alpha x}{y} + \cfrac{\beta\beta x : y}{\gamma - \frac{\beta x}{y} + \text{etc.}}}$$

quae a fractionibus partialibus liberata dat

$$\frac{x}{sy} = \alpha + \cfrac{\alpha\alpha x}{\beta y - \alpha x + \cfrac{\beta\beta x y}{\gamma y - \beta x + \cfrac{\gamma\gamma x y}{\delta y - \gamma x + \text{etc.}}}}$$

vnde nascitur sequens

Theorema IV.

§. 34. *Si proposita fuerit huiusmodi series infinita:*

$$s = \frac{x}{\alpha y} - \frac{xx}{\beta yy} + \frac{x^3}{\gamma y^3} - \frac{x^4}{\delta y^4} + \text{etc.}$$

ex ea formari poterit ista fractio continua:

$$\frac{x}{s} = \alpha y + \cfrac{\alpha\alpha x y}{\beta y - \alpha x + \cfrac{\beta\beta x y}{\gamma y - \beta x + \cfrac{\gamma\gamma x y}{\delta y - \gamma x + \cfrac{\delta\delta x y}{\epsilon y - \delta x + \text{etc}}}}}$$

§. 35. Cum sit

$$l\left(1 + \frac{x}{y}\right) = \frac{x}{y} - \frac{xx}{2yy} + \frac{x^3}{3y^3} - \frac{x^4}{4y^4} + \text{etc.}$$

posito

posito $s = l(1+\frac{x}{y})$ erit

$$\alpha = 1,\ \beta = 2,\ \gamma = 3,\ \delta = 4,\ \text{etc.}$$

hincque nafcetur ifta fractio continua:

$$\frac{x}{l(1+\frac{x}{y})} = y + \cfrac{1\,x.y}{2y - x + \cfrac{4xy}{3y - 2x + \cfrac{9xy}{4y - 3x + \text{etc.}}}}$$

§. 36. Cum arcus cuius tangens t hac ferie exprimatur:

$$\text{A. tang.}\, t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \text{etc.} \quad \text{erit}$$

$$t\,\text{A tang.}\, t = \frac{tt}{1} - \frac{t^4}{3} + \frac{t^6}{5} - \frac{t^8}{7} + \text{etc.}$$

Nunc ponatur $tt = \frac{x}{y}$, ita vt fit $t = \sqrt{\frac{x}{y}}$, fietque

$$\sqrt{\frac{x}{y}}.\,\text{A tang.}\sqrt{\frac{x}{y}} = \frac{x}{y} - \frac{xx}{3yy} + \frac{x^3}{5y^3} - \frac{x^4}{7y^4} + \text{etc.}$$

Hinc ergo eft $s = \sqrt{\frac{x}{y}}.\,\text{A tang.}\sqrt{\frac{x}{y}}$, tum vero

$$\alpha = 1,\ \beta = 3,\ \gamma = 5,\ \delta = 7,\ \text{etc.},$$

quare fractio continua hinc nata erit

$$\frac{\sqrt{xy}}{\text{A tang.}\sqrt{\frac{x}{y}}} = y + \cfrac{xy}{3y - x + \cfrac{9xy}{5y - 3x + \cfrac{25xy}{7y - 5x + \text{etc.}}}}$$

Veluti fi fuerit $x = 1$ et $y = 3$, ob A tang. $\frac{1}{\sqrt{3}} = \frac{\pi}{6}$, habebitur ifta fractio continua:

6 √3

$$\frac{6\sqrt{3}}{\pi} = 3 + \cfrac{1.3}{8 + \cfrac{3.9}{12 + \cfrac{3.25}{16 + \text{etc.}}}}$$

§. 37. Quod si in casu Theorematis loco litterarum α, β, γ, δ, etc. scribamus fractiones

$$\frac{\alpha}{a},\ \frac{\beta}{b},\ \frac{\gamma}{c},\ \frac{\delta}{d},\ \text{etc.}$$

vt habeamus hanc seriem:

$$s = \frac{ax}{\alpha y} - \frac{bxx}{\beta yy} + \frac{cx^3}{\gamma y^3} - \frac{dx^4}{\delta y^4} + \text{etc.}$$

fractio continua hinc formata ita se habebit:

$$\frac{x}{s} = \frac{\alpha}{a}y + \cfrac{\alpha\alpha xy : aa}{\frac{\beta}{b}y - \frac{\alpha}{a}x + \cfrac{\beta\beta xy : bb}{\frac{\gamma}{c}y - \frac{\beta}{b}x + \cfrac{\gamma\gamma xy : cc}{\frac{\delta}{d}y - \frac{\gamma}{c}x + \text{etc.}}}}$$

Hic iam primo vtrinque multiplicetur per a, deinde primae fractionis numerator et denominator multiplicentur per ab, secundae per bc, tertiae per cd, etc. et fractio continua hanc induet formam:

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha\alpha bxy}{a\beta y - \alpha bx + \cfrac{\beta\beta acxy}{b\gamma y - c\beta x + \cfrac{\gamma\gamma bdxy}{c\delta y - d\gamma x + \text{etc.}}}}$$

vnde operae pretium erit sequens apponere

Theo-

Theorema V.

§. 38. *Si proposita fuerit series infinita huius formae:*

$$s = \frac{ax}{\alpha y} - \frac{bxx}{\beta\gamma y} + \frac{cx^3}{\gamma y^3} - \frac{dx^4}{\delta y^4} + \text{etc.}$$

inde formabitur sequens fractio continua:

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha\alpha bxy}{\alpha\beta y - \alpha bx + \cfrac{\beta\beta acxy}{b\gamma y - c\beta x + \cfrac{\gamma\gamma bdxy}{c\delta y - d\gamma x + \text{etc.}}}}$$

Problema III.

Propositam hanc seriem infinitam:

$$s = \frac{1}{\alpha} - \frac{1}{\alpha\beta} + \frac{1}{\alpha\beta\gamma} - \frac{1}{\alpha\beta\gamma\delta} + \text{etc.}$$

in fractionem continuam conuertere.

Solutio.

§. 39. Ex serie proposita formemus sequentes series:

$$t = \frac{1}{\beta} - \frac{1}{\beta\gamma} + \frac{1}{\beta\gamma\delta} - \frac{1}{\beta\gamma\delta\varepsilon} + \text{etc.},$$

$$u = \frac{1}{\gamma} - \frac{1}{\gamma\delta} + \frac{1}{\gamma\delta\varepsilon} - \frac{1}{\gamma\delta\varepsilon\zeta} + \text{etc.}$$

atque habebimus

$$s = \frac{1-t}{\alpha},\; t = \frac{1-u}{\beta},\; u = \frac{1-v}{\gamma} \text{ etc.}$$

hinc igitur deducimus

$$\frac{1}{s} = \frac{\alpha}{1-t} = \alpha + \frac{\alpha t}{1-t} = \alpha + \cfrac{\alpha}{-1+\cfrac{1}{t}}.$$

Simili autem modo erit

$$\frac{1}{t} = \beta + \frac{\beta}{-1 + \frac{1}{u}}, \quad \frac{1}{u} = \gamma + \frac{\gamma}{-1 + \frac{1}{v}}, \text{ etc.}$$

quare posterioribus valoribus in prioribus substitutis obtinebitur ista fractio continua:

$$\frac{1}{s} = \alpha + \cfrac{\alpha}{\beta - 1 + \cfrac{\beta}{\gamma - 1 + \cfrac{\gamma}{\delta - 1 + \text{etc.}}}}$$

vnde deducimus sequens Theorema.

Theorema VI.

§. 40. *Si proposita fuerit huiusmodi series infinita*

$$s = \frac{1}{\alpha} - \frac{1}{\alpha\beta} + \frac{1}{\alpha\beta\gamma} - \frac{1}{\alpha\beta\gamma\delta} + \text{etc.}$$

exinde formari poterit haec fractio continua:

$$\frac{1}{s} = \alpha + \cfrac{\alpha}{\beta - 1 + \cfrac{\beta}{\gamma - 1 + \cfrac{\gamma}{\delta - 1 + \text{etc.}}}}$$

§. 41. Si e denotet numerum cuius logarithmus hyperbolicus est vnitas, notum est esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \text{etc. siue}$$

$$\frac{e-1}{e} = \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \text{etc.}$$

Hic igitur fit $s = \frac{e-1}{e}$, tum vero

$$\alpha = 1, \beta = 2, \gamma = 3, \delta = 4, \text{ etc.}$$

quare fractio continua hinc oriunda est

$\frac{e}{e-1}$

$$\frac{e}{e-1} = 1 + \cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\text{etc.}}}}$$

§. 42. Cum igitur sit

$$\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\text{etc.}}}}} = \frac{1}{e-1}$$

haud difficulter autem demonstrari queat, si fuerit

$$\cfrac{a}{a+\cfrac{b}{b+\cfrac{c}{c+\text{etc.}}}} = s$$

eum fore

$$a + \cfrac{a}{b+\cfrac{b}{c+\cfrac{c}{d+\text{etc.}}}} = \frac{s}{1-s}$$

pro nostro casu erit

$$s = \frac{1}{e-1},\ a = 1;\ b = 2,\ c = 3;\ \text{etc.}$$

quibus valoribus substitutis fiet

$$1 + \cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\text{etc.}}}} = \frac{1}{e-2}$$

§. 43. Quodsi in serie Theorematis VI loco literarum α, β, γ, δ, etc. scribantur fractiones

$\frac{\alpha}{a}$, $\frac{\beta}{b}$, $\frac{\gamma}{c}$, $\frac{\delta}{d}$ etc., vt sit

$$s = \frac{a}{\alpha} - \frac{ab}{\alpha\beta} + \frac{abc}{\alpha\beta\gamma} - \frac{abcd}{\alpha\beta\gamma\delta} + \text{etc.}$$

fractio continua hinc nata erit

$$\frac{1}{s} = \frac{\alpha}{a} + \cfrac{\alpha : a}{\frac{\beta}{b} - 1 + \cfrac{\beta : b}{\frac{\gamma}{c} - 1 + \cfrac{\gamma : c}{\frac{\delta}{d} - 1 + \text{etc.}}}}$$

Quodsi iam primo multiplicetur vtrinque per a, tum vero prima fractio supra et infra per b, secunda per c, tertia per d, etc., orietur ista forma:

$$\frac{a}{s} = \alpha + \cfrac{\alpha b}{\beta - b + \cfrac{\beta c}{\gamma - c + \cfrac{\gamma d}{\delta - d + \text{etc.}}}}$$

quod sequenti Theorema includatur

Theorema VII.

§. 44. *Si proposita fuerit huiusmodi series infinita:*

$$s = \frac{a}{\alpha} - \frac{ab}{\alpha\beta} + \frac{abc}{\alpha\beta\gamma} - \frac{abcd}{\alpha\beta\gamma\delta} + \text{etc.}$$

inde deducitur haec fractio continua:

$$\frac{a}{s} = \alpha + \cfrac{\alpha b}{\beta - b + \cfrac{\beta c}{\gamma - c + \cfrac{\gamma d}{\delta - d + \text{etc.}}}}$$

§. 45.

§. 45. Applicemus hoc ad sequentem seriem infinitam:

$$s = \frac{1}{2} - \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} - \frac{1.3.5.7}{2.4.6.8} + \text{etc.}$$

cuius summam constat esse $s = \frac{\sqrt{2}-1}{\sqrt{2}}$; tum igitur erit

$$a = 1,\ b = 3,\ c = 5,\ d = 7,\ \text{etc.}$$
$$\alpha = 2,\ \beta = 4,\ \gamma = 6,\ \delta = 8,\ \text{etc.}$$

fractio ergo continua hinc nata erit

$$\frac{\sqrt{2}}{\sqrt{2}-1} = 2 + \cfrac{2.3}{1 + \cfrac{4.5}{1 + \cfrac{6.7}{1 + \text{etc.}}}}$$

Si vtrinque vnitas auferatur erit

$$\frac{1}{\sqrt{2}-1} = 1 + \cfrac{2.3}{1 + \cfrac{4.5}{1 + \cfrac{6.7}{1 + \text{etc.}}}}$$

vnde deducitur

$$\sqrt{2} = 1 + \cfrac{1.1}{1 + \cfrac{2.3}{1 + \cfrac{4.5}{1 + \cfrac{6.7}{1 + \text{etc.}}}}}$$

Problema IV.

Propositam seriem infinitam huius formae:

$$s = \frac{x}{\alpha} - \frac{xx}{\alpha\beta} + \frac{x^3}{\alpha\beta\gamma} - \frac{x^4}{\alpha\beta\gamma\delta} + \text{etc.}$$

in fractionem continuam conuertere.

 Solutio

Solutio.

§. 46. Statuamus vt hactenus

$$t = \frac{x}{\beta} - \frac{x\,x}{\beta\gamma} + \frac{x^3}{\beta\gamma\delta} - \frac{x^4}{\beta\gamma\delta\varepsilon} + \text{etc. et}$$

$$u = \frac{x}{\gamma} - \frac{x\,x}{\gamma\delta} + \frac{x^3}{\gamma\delta\varepsilon} - \frac{x^4}{\gamma\delta\varepsilon\zeta} + \text{etc.}$$

ita vt sit $s = \frac{x - t\,x}{\alpha}$, vnde fit $\frac{x}{s} = \frac{\alpha}{1-t} = \alpha + \frac{\alpha t}{1-t}$. Est vero

$$\frac{\alpha t}{1-t} = \frac{\alpha}{-1+\frac{1}{t}} = \frac{\alpha x}{-x+\frac{x}{t}},$$

sicque erit

$$\frac{x}{s} = \alpha + \frac{\alpha x}{-x+\frac{x}{t}}.$$

Simili igitur modo reperietur

$$\frac{x}{t} = \beta + \frac{\beta x}{-x+\frac{x}{u}},\quad \frac{x}{u} = \gamma + \frac{\gamma x}{-x+\frac{x}{v}},\ \text{etc.}$$

Quodsi ergo hi valores continuo in praecedentibus substituantur, obtinebitur sequens fractio continua:

$$\frac{x}{s} = \alpha + \cfrac{\alpha x}{\beta - x + \cfrac{\beta x}{\gamma - x + \cfrac{\gamma x}{\delta - x + \text{etc.}}}}$$

hinceque nascitur

Theorema VIII.

§. 47. *Si proposita fuerit huiusmodi series infinita:*

$$s = \frac{x}{\alpha} - \frac{xx}{\alpha\beta} + \frac{x^3}{\alpha\beta\gamma} - \frac{x^4}{\alpha\beta\gamma\delta} + \text{etc.}$$

inde formabitur sequens fractio continua:

$$\frac{x}{s} = \alpha + \cfrac{\alpha x}{\beta - x + \cfrac{\beta x}{\gamma - x + \cfrac{\gamma x}{\delta - x + \text{etc.}}}}$$

§. 48. Quod si hic loco x scribamus $\frac{x}{y}$, vt habeamus hanc seriem:

$$s = \frac{x}{\alpha y} - \frac{xx}{\alpha\beta yy} + \frac{x^3}{\alpha\beta\gamma y^3} - \frac{x^4}{\alpha\beta\gamma\delta y^4} + \text{etc.}$$

hinc nascetur sequens fractio continua;

$$\frac{x}{s} = \alpha y + \cfrac{\alpha xy}{\beta y - x + \cfrac{\beta xy}{\gamma y - x + \cfrac{\gamma xy}{\delta y - x + \text{etc.}}}}$$

§. 49. Sumamus $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\delta = 4$ etc. vt sit

$$s = \frac{x}{y} - \frac{xx}{1.2.yy} + \frac{x^3}{1.2.3\,y^3} - \frac{x^4}{1.2.3.4\,y^4} + \text{etc.}$$

vbi igitur est $s = 1 - e^{\frac{-x}{y}}$, hincque formabitur ista fractio continua:

$$\frac{x}{1 - e^{\frac{-x}{y}}} = y + \cfrac{xy}{2y - x + \cfrac{2xy}{3y - x + \cfrac{3xy}{4y - x + \text{etc.}}}}$$

=

$= \frac{x e^{\frac{x}{y}}}{e^{\frac{x}{y}} - 1}$, vnde obtinebuntur sequentes formulae speciales: sumendo $x = 1$ et loco y successiue numeros 1, 2, 3, 4, 5 etc.

$$\frac{e}{e-1} = 1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \text{etc.}}}}$$

$$\frac{\sqrt{e}}{\sqrt{e}-1} = 2 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \text{etc.}}}}$$

$$\frac{\sqrt[3]{e}}{\sqrt[3]{e}-1} = 3 + \cfrac{3}{5 + \cfrac{6}{8 + \cfrac{9}{11 + \text{etc.}}}}$$

$$\frac{\sqrt[4]{e}}{\sqrt[4]{e}-1} = 4 + \cfrac{4}{7 + \cfrac{8}{11 + \cfrac{12}{15 + \text{etc.}}}}$$

Problema V.

Si in genere proposita fuerit series huius formae:

$$s = \frac{a x}{\alpha y} - \frac{a b x^2}{\alpha \beta y^2} + \frac{a b c x^3}{\alpha \beta \gamma y^3} - \frac{a b c d x^4}{\alpha \beta \gamma \delta y^4} + \text{etc.}$$

eam

eam in fractionem continuam conuertere

Solutio.

§. 50. Ex ſerie propoſita formemus ſequentes:

$$t = \frac{bx}{\beta y} - \frac{bcxx}{\beta\gamma yy} + \frac{bcdx^3}{\beta\gamma\delta y^3} - \frac{bcdex^4}{\beta\gamma\delta\varepsilon y^4} + \text{etc. et}$$

$$u = \frac{cx}{\gamma y} - \frac{cdxx}{\gamma\delta yy} + \frac{cdex^3}{\gamma\delta\varepsilon y^3} - \frac{cdefx^4}{\gamma\delta\varepsilon\zeta y^4} + \text{etc.}$$

ita vt ſit $s = \frac{ax}{\alpha y}(1 - t)$ hincque

$$\frac{ax}{s} = \frac{\alpha y}{1-t} = \alpha y + \frac{\alpha y t}{1-t}.$$

Eſt vero

$$\frac{\alpha y t}{1-t} = \cfrac{\alpha y}{-1 + \cfrac{1}{t}} = \cfrac{\alpha bxy}{-bx + \cfrac{bx}{t}}$$

vnde fit

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{-bx + \cfrac{bx}{t}};$$

ſimili igitur modo ex relatione $\frac{bx}{\beta y}(1-u)$ fiet

$$\frac{bx}{t} = \beta y + \cfrac{\beta cxy}{-cx + \cfrac{cx}{u}}$$

ſicque porro. Quare his valoribus continuo ſubſtitutis orietur iſta fractio continua:

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{\beta y - bx + \cfrac{\beta cxy}{\gamma y - cx + \cfrac{\gamma dxy}{\delta y - dx + \text{etc.}}}}$$

hinc ſequens

Theorema IX.

§. 51. *Si proposita fuerit ista series generalis:*

$$s = \frac{ax}{\alpha y} - \frac{abxx}{\alpha\beta yy} + \frac{abcx^3}{\alpha\beta\gamma y^3} - \frac{abcdx^4}{\alpha\beta\gamma\delta y^4} + \text{etc.}$$

inde formabitur fractio continua

$$\frac{ax}{s} = \alpha y + \cfrac{\alpha bxy}{\beta y - bx + \cfrac{\beta cxy}{\gamma y - cx + \cfrac{\gamma dxy}{\delta y - dx + \text{etc.}}}}$$

§. 52. Vt hoc Theorema per exemplum notatu dignum illustremus, consideremus hanc formulam integralem:

$$Z = \int z^{m-1} dz (1 + z^n)^{\frac{k}{n} - 1}$$

quod integrale ita sumatur vt euanescat posito $z = 0$. statuamus $Z = v(1 + z^n)^{\frac{k}{n}}$, eritque differentiando

$$dZ = z^{m-1} dz (1 - z^n)^{\frac{k}{n} - 1} = dv (1 + z^n)^{\frac{k}{n}}$$
$$+ kvz^{n-1} dz (1 + z^n)^{\frac{k}{n} - 1},$$

quae aequatio per $(1 + z^n)^{\frac{k}{n} - 1}$ diuisa praebet

$$z^{m-1} dz = dv(1 + z^n) + kvz^{n-1} dz, \text{ ideoque}$$
$$\frac{dv}{dz}(1 + z^n) + kvz^{n-1} - z^{m-1} = 0.$$

§. 53. Quoniam sumto z infinite paruo fit

$$Z = \frac{z^m}{m} = v,$$

inde discimus, quantitatem v per eiusmodi seriem infinitam exprimi, cuius primus terminus sit potestas z^m; in sequentibus

tibus autem terminis exponentes ipsius z continuo numero n augeri, quamobrem pro v fingamus sequentem seriem infinitam:

$$v = Az^m - Bz^{m+n} + Cz^{m+2n} - Dz^{m+3n} + \text{etc.}$$

quem valorem in aequatione differentiali substituamus, similesque potestates ipsius z sibi subscribamus sequendi modo:

$$\frac{dv}{dz} = mAz^{m-1} - (m+n)Bz^{m+n-1} + (m+2n)Cz^{m+2n-1} - (m+3n)Dz^{m+3n-1} + \text{etc.}$$

$$\begin{array}{llll} \frac{z^n dv}{dz} = & +mA & -(m+n)B & +(m+2n)C - \text{etc.} \\ +kvz^{n-1} = & +kA & -kB & +kC \\ -z^{m-1} = & -1. & & \end{array}$$

§. 54. Quod si nunc singulae ipsius z potestates seorsim ad nihilum redigantur, obtinebuntur sequentes valores:

$$\begin{array}{ll} mA - 1 = 0 & \text{ergo } A = \frac{1}{m} \\ -(m+n)B + (m+k)A = 0 & \text{ergo } B = \frac{(m+k)A}{m+n} \\ (m+2n)C - (m+n+k)B = 0 & \text{ergo } C = \frac{(m+n+k)B}{m+2n} \\ -(m+3n)D + (m+2n+k)C = 0 & \text{ergo } D = \frac{(m+2n+k)C}{m+3n} \\ \text{etc} & \text{etc.} \end{array}$$

§. 55. Substituamus igitur hos valores inuentos ac pro v reperiemus sequentem seriem infinitam:

$$v = \frac{z^m}{m} - \frac{m+k}{m(m+n)}z^{m+n} + \frac{(m+k)(m+n+k)}{m(m+n)(m+2n)}z^{m+2n}$$
$$- \frac{(m+k)(m+n+k)(m+2n+k)}{m(m+n)(m+2n)(m+3n)}z^{m+3n} \text{ etc.}$$

 quam

quam feriem, vt ad formam noſtri theorematis reducamus; hoc modo repraeſentemus:

$$v = \frac{z^{m-n}}{m}\left(z^n - \frac{m+k}{m+n}z^{2n} + \frac{(m+k)(m+n+k)}{(m+n)(m+2n)}z^{3n} - \frac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n} \text{ etc.}\right)$$

§. 56. Cum iam ſit

$$Z = \int z^{m-1}dz(1+z^n)^{\frac{k}{n}} - 1, \text{ ſtatuamus}$$

$$V = \frac{mZ}{z^{m-n}(1+z^n)^{\frac{k}{n}}}, \text{ vt fiat}$$

$$V = z^n - \frac{m+k}{m+n}z^{2n} + \frac{(m+k)(m+n+k)}{(m+n)(m+2n)}z^{3n} - \frac{(m+k)(m+n+k)(m+2n+k)}{(m+n)(m+2n)(m+3n)}z^{4n} + \text{etc.}$$

erit igitur V functio ipſius z per integrationem formulae differentialis eruenda, quae ergo pro quouis valore ipſius z determinatum adipiſcetur valorem, ſiquidem integrale ita capi ponimus, vt euaneſcat facto $z = 0$. Quo igitur facilius iſtos valores ipſius v aſſignare queamus, quando variabili z valores fracti tribuantur, ſtatuamus in genere $z^n = \frac{x}{y}$, ita vt hanc formam nanciſcamur:

$$V = \frac{x}{y} - \frac{(m+k)xx}{(m+n)yy} + \frac{(m+k)(m+n+k)x^3}{(m+n)(m+2n)y^3} - \frac{(m+k)(m+n+k)(m+2n+k)x^4}{(m+n)(m+2n)(m+3n)y^4} + \text{etc.}$$

quae feries cum noſtro Theoremate collata praebet $s = V$; tum vero

$$a = 1,\ b = m+k,\ c = m+n+k, \text{ etc.}$$

$$\alpha = 1,\ \beta = m+n,\ \gamma = m+2n, \text{ etc.}$$

§. 57. His notatis formula integralis aſſumta ſequentem nobis ſuppeditat fractionem continuam:

$\frac{x}{V} =$

$$\frac{x}{V} = y + \cfrac{(m+k)xy}{(m+n)y - (m+k)x + \cfrac{(m+n)(m+n+k)xy}{(m+2n)y - (m+n+k)x + \cfrac{(m+2n)(m+2n+k)xy}{(m+3n)y - (m+2n+k)x + \text{etc.}}}}$$

cuius valor erit $\frac{x z^{m-n}(1+z^n)^{\frac{k}{n}}}{mZ}$.

§. 58. Exempli gratia sumamus hanc formam:

$$Z = \int \frac{dz}{\sqrt{(1+zz)}} = l(z + \sqrt{(1+zz)}),$$

erit igitur $m = 1$, $n = 2$, $k = 1$, fietque $V = \frac{z\,l(z+\sqrt{(1+zz)})}{\sqrt{(1+zz)}}$ qui valor aequatur huic seriei:

$$zz - \tfrac{2}{3}z^4 + \tfrac{2.4}{3.5}z^6 - \tfrac{2.4.6}{3.5.7}z^8 + \text{etc.}$$

quae, si iam loco zz scribamus $\frac{x}{y}$, fiet

$$V = \frac{\sqrt{x}}{\sqrt{(x+y)}}\, l\left(\frac{\sqrt{x}+\sqrt{(x+y)}}{\sqrt{y}}\right)$$
$$= \frac{x}{y} - \tfrac{2}{3}.\frac{xx}{yy} + \tfrac{2.4}{3.5}.\frac{x^3}{y^3} - \tfrac{2.4.6}{3.5.7}.\frac{x^4}{y^4} + \text{etc.}$$

quare fractio continua hinc formata erit

$$\frac{\sqrt{x(x+y)}}{l\frac{(\sqrt{x}+\sqrt{(x+y)})}{\sqrt{y}}} = y + \cfrac{1.2xy}{3y - 2x + \cfrac{3.4xy}{5y - 4x + \cfrac{5.6xy}{7y - 6x + \cfrac{7.8xy}{9y - 8x + \text{etc.}}}}}$$

§. 59. Quodsi ergo sumamus $x = 1$ et $y = 1$, habebimus istam fractionem continuam:

$$\frac{\sqrt{2}}{l(1+\sqrt{2})} = 1 + \cfrac{1.2}{1 + \cfrac{3.4}{1 + \cfrac{5.6}{1 + \text{etc.}}}}$$

ipſa ſerie infinita exiſtente

$$\frac{l(1+\sqrt{2})}{\sqrt{2}} = 1 - \frac{2}{1.3} + \frac{2.4}{3.5} - \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} - \text{etc.}$$

Sin autem manente $x = 1$ ſumamus $y = 2$, ſeries infinita erit

$$\frac{1}{2} - \frac{2}{3}\cdot\frac{1}{4} + \frac{2.4}{3.5}\cdot\frac{1}{8} - \frac{2.4.6}{3.5.7}\cdot\frac{1}{16} + \text{etc.}$$

fractio vero continua haec:

$$\frac{\sqrt{3}}{l\frac{1+\sqrt{3}}{\sqrt{2}}} = 2 + \cfrac{1.2.2}{4 + \cfrac{3.4.2}{6 + \cfrac{5.6.2}{8 + \text{etc.}}}}$$

vnde ſequens forma deducitur:

$$\frac{\sqrt{3}}{2\,l\frac{1+\sqrt{3}}{\sqrt{2}}} = 1 + \cfrac{1}{2 + \cfrac{6}{3 + \cfrac{15}{4 + \cfrac{28}{5 + \text{etc.}}}}}$$

vbi numeratores ſunt numeri trigonales alternatim ſumti.

§. 60. Euoluamus adhuc caſum, quo $x = 1$ et $y = 3$, quoniam irrationalitas hoc modo tolletur; erit autem hoc caſu

$$\frac{1}{4} l 3 = \frac{1}{3} - \frac{1}{3}\cdot\frac{1}{5} + \frac{1}{3\cdot 3}\cdot\frac{1}{7} - \frac{1}{3\cdot 3\cdot 3}\cdot\frac{1}{9} + \frac{1}{3\cdot 3\cdot 3\cdot 3}\cdot\frac{1}{11} + \text{etc.}$$

fractio autem continua hinc nata erit

$$\frac{4}{l 3} = 3 + \cfrac{1.\,2.\,3}{7 + \cfrac{3.\,4.\,3}{11 + \cfrac{5.\,6.\,3}{15 + \cfrac{7.\,8.\,3}{19 + \text{etc.}}}}}$$

§. 61. Quoniam igitur hic nouam plane methodum aperui, series quascunque infinitas in fractiones continuas transformandi, merito equidem mihi videor doctrinam fractionum continuarum haud mediocriter locupletasse. His igitur tantum subiungam theorema notatu dignissimum, quo supra § 42. fractionem

$$\cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \text{etc.}}}}} = \frac{1}{e-1}$$

transformauimus in hanc:

$$1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \text{etc.}}}} = \frac{1}{e-2}$$

quod multo latius patens ita se habet

Theo-

Theorema

§. 62. *Si fuerit*

$$s = \cfrac{aA}{\alpha A + \cfrac{bB}{\beta B + \cfrac{cC}{\gamma C + \text{etc.}}}}$$

erit

$$\frac{bs}{a - \alpha s} = \beta A + \cfrac{cA}{\gamma B + \cfrac{dB}{\delta C + \cfrac{eC}{\varepsilon D + \text{etc.}}}}$$

Demonstratio

Cum enim sit

$$s = \cfrac{aA}{\alpha A + \cfrac{bB}{\beta B + \cfrac{cC}{\gamma C + \text{etc.}}}}$$

si primam fractionem per A, secundam per B, tertiam per C, etc. deprimamus, prodibit

$$s = \cfrac{a}{\alpha + \cfrac{b:A}{\beta + \cfrac{c:B}{\gamma + \cfrac{d:C}{\delta + \text{etc.}}}}}$$

Nunc huius formae secundam fractionem supra et infra per A, tertiam per B, quartam per C multiplicemus et ita porro, et nanciscemur hanc formam:

$s =$

$$s = \cfrac{a}{\alpha + \cfrac{b}{\beta A + \cfrac{cA}{\gamma B + \cfrac{dB}{\delta C + \text{etc.}}}}}$$

quare si statuamus

$$t = \beta A + \cfrac{cA}{\gamma B + \cfrac{dB}{\delta C + \text{etc.}}}$$

erit

$$s = \frac{a}{\alpha + \frac{b}{t}} = \frac{at}{\alpha t + b}$$

vnde reperitur $t = \frac{bs}{a - \alpha s}$ q. e. d.

METHODVS INVENIENDI FORMVLAS INTEGRALES,

QVAE

CERTIS CASIBVS DATAM INTER SE TENEANT RATIONEM,

VBI SIMVL METHODVS TRADITVR FRACTIONES CONTINVAS SVMMANDI.

§. 1.

Quemadmodum in feriebus recurrentibus quilibet terminus ex vno pluribusue praecedentibus fecundum legem quandam conftantem determinatur, ita hic eiusmodi feries fum confideraturus, in quibus quilibet terminus ex vno pluribusue praecedentibus fecundum quampiam legem variabilem determinatur. Quoniam autem in talibus feriebus formula generalis fingulos terminos exprimens plerumque non eft algebraica, fed transcendens, fingulos terminos per formulas integrales exhiberi conueniet, quae vt valores determinatos praebeant, poft integrationem quantitati variabili valorem determinatum tribui affumo, ita vt finguli termini prodeant quantitates determinatae; atque nunc quaeftio principalis huc redit, quemadmodum iftae formulae integrales debeant effe comparatae, vt quilibet terminus fecundum datam legem ex vno pluribusue praecedentibus determinetur.

§. 2.

§. 2. Quod quo clarius perſpiciatur, contemplemur ſeriem notiſſimam harum formularum integralium:

$$\int\frac{dx}{\sqrt{(1-xx)}},\ \int\frac{xx\,dx}{\sqrt{(1-xx)}},\ \int\frac{x^4\,dx}{\sqrt{(1-xx)}},\ \int\frac{x^6\,dx}{\sqrt{(1-xx)}},\ \text{etc.}$$

quae ſi ſingulae ita integrentur, vt euaneſcant poſito $x=0$, tum vero variabili x tribuatur valor $=1$, quilibet terminus a praecedente ita pendet, vt ſit

$$\int\frac{xx\,dx}{\sqrt{(1-xx)}}=\tfrac{1}{2}\int\frac{dx}{\sqrt{(1-xx)}},$$
$$\int\frac{x^4\,dx}{\sqrt{(1-xx)}}=\tfrac{3}{4}.\int\frac{xx\,dx}{\sqrt{(1-xx)}},$$
$$\int\frac{x^6\,dx}{\sqrt{(1-xx)}}=\tfrac{5}{6}.\int\frac{x^4\,dx}{\sqrt{(1-xx)}},$$

atque in genere

$$\int\frac{x^n\,dx}{\sqrt{(1-xx)}}=\frac{n-1}{n}\int\frac{x^{n-2}\,dx}{\sqrt{(1-xx)}}.$$

Vnde patet, hanc formulam generalem ſpectari poſſe tanquam terminum generalem illius ſeriei, atque quemlibet terminum ex praecedente oriri, ſi iſte multiplicetur per $\frac{n-1}{n}$.

§. 3. Ad ſimilitudinem igitur huius caſus ſeriem formularum integralium ita in genere conſtituamus,

$$\int dv,\ \int x\,dv.\ \int xx\,dv,\ \int x^3\,dv,\ \int x^4\,dv,\ \text{etc.}$$

ita vt terminus indici n reſpondens ſit $\int x^{n-1}\,dv$, quae ſingula integralia ita accipi ſumamus, vt euaneſcant poſito $x=0$; poſt integrationem autem quantitati variabili x tribuamus quempiam valorem conſtantem, veluti $x=1$, vel alio cuipiam numero. Quibus poſitis quaeſtio huc redit, qualis pro v aſſumi debeat functio ipſius x, vt quilibet terminus per vnum, vel duos pluresue praecedentes, ſecundum legem quandam datam vtcunque variabilem, ſiue ab indice n pendentem, determinetur; vbi quidem imprimis eo erit reſpiciendum, ad quot dimenſiones index n in ſcala relationis propoſita aſcendat: plerumque autem non vltra pri-

 mam

mam dimensionem assurgere erit opus. Hinc igitur sequentia Problemata pertractemus.

Problema I.

Inuenire functionem v, *vt ista relatio inter binos terminos sibi succedentes locum habeat:*

$$\int x^n\, dv = \frac{\alpha n + a}{\beta n + b} \int x^{n-1}\, dv.$$

Solutio.

§. 4. Requiritur igitur hic vt sit

$$(\alpha n + a) \int x^{n-1}\, dv = (\beta n + b) \int x^n\, dv$$

si scilicet post integrationem variabili x certus valor tribuatur. Quoniam igitur ista conditio tum demum locum habere debet, postquam variabili x iste valor constans fuerit datus, ponamus in genere, dum x est variabilis, hanc aequationem locum habere:

$$(\alpha n + a) \int x^{n-1}\, dv = (\beta n + b) \int x^n\, dv + V,$$

quantitatem autem V ita esse comparatam, vt euanescat postquam variabili ille valor determinatus fuerit assignatus. Praeterea vero, quia ambo integralia ita capi assumimus, vt euanescant posito $x = 0$, necesse est vt etiam ista quantitas V eodem quoque casu euanescat.

§. 5. Quoniam haec aequalitas subsistere debet pro omnibus indicibus n, quos quidem semper vt positiuos spectamus, facile intelligitur, quantitatem istam V factorem habere debere x^n; quo pacto iam isti conditioni satisfit, vt posito $x = 0$ etiam fiat $v = 0$. Quamobrem statuamus $V = x^n Q$; vbi Q denotet functionem ipsius x proposito accom-

accommodatam, et quam simul ita comparatam esse desideramus, vt euanescat si ipsi x certus quidam valor tribuatur.

§. 6. Cum igitur esse debeat

$$(\alpha n + a)\int x^{n-1} dv = (\beta n + b)\int x^n dv + x^n Q,$$

differentietur ista aequatio, ac differentiali per x^{n-1} diuiso peruenietur ad hanc aequationem differentialem:

$$(\alpha n + a) dv = (\beta n + b) x\, dv + n Q dx + x dQ,$$

quae cum subsistere debeat pro omnibus valoribus ipsius n, termini ista littera affecti seorsim se tollere debent, vnde nanciscimur has duas aequalitates:

$$\text{I. } (\alpha - \beta x) dv = Q dx \quad \text{et} \quad \text{II. } (a - bx) dv = x dQ.$$

Ex priore fit $dv = \frac{Q dx}{\alpha - \beta x}$; ex altera vero $dv = \frac{x dQ}{a - bx}$; qui duo valores inter se aequati suppeditant hanc aequationem: $\frac{dQ}{Q} = \frac{dx}{x} \cdot \frac{a - bx}{\alpha - \beta x}$, quae aequatio resoluitur in has partes

$$\frac{dQ}{Q} = \frac{a}{\alpha} \cdot \frac{dx}{x} + \frac{a\beta - b\alpha}{\alpha} \cdot \frac{dx}{\alpha - \beta x},$$

cuius ergo integrale erit

$$lQ = \frac{a}{\alpha} lx - \frac{a\beta - b\alpha}{\alpha\beta} l(\alpha - \beta x)$$

vnde deducitur

$$Q = C\, x^{\frac{a}{\alpha}} \cdot (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

§. 7. Ex hoc valore pro Q inuento statim patet eum euanescere casu $x = \frac{\alpha}{\beta}$, si modo fuerit $\frac{b\alpha - a\beta}{\alpha\beta} > 0$; sin autem secus eueniat, non patet quomodo haec quantitas vllo casu euanescere queat. Inuento autem hoc valore Q, inde reperietur

$$dv = C\, x^{\frac{a}{\alpha}} dx\, (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1}$$

 hinc

hincque nostrae seriei terminus indici n respondens erit

$$\int x^{n-1}dv = C\int x^{n+\frac{a}{\alpha}-1}dx(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}-1},$$

tum vero erit

$$V = Cx^{n+\frac{a}{\alpha}}(\alpha-\beta x)^{\frac{b\alpha-a\beta}{\alpha\beta}}.$$

Vbi res imprimis eo redit, vt ista quantitas praeter casum $x=0$ insuper alio casu euanescat.

Corollarium 1.

§. 8. Hic duo casus occurrunt, qui peculiarem euolutionem postulant; prior est, quo $\alpha=0$; tum autem inchoandum erit ab aequatione $\frac{dQ}{Q}=-\frac{(a-bx)dx}{\beta xx}$, vnde integrando elicitur $lQ=\frac{a}{\beta x}+\frac{b}{\beta}lx$, hinque sumendo e pro numero cuius logarithmus hyperbolicus $=1$, colligitur

$$Q=e^{\frac{a}{\beta x}}.x^{\frac{b}{\beta}}$$

quae formula in nihilum abire nequit, nisi fiat $\frac{a}{\beta x}=-\infty$, ideoque $x=0$, sicque non duo haberentur casus, quibus fieret $V=0$, cum tamen duo desiderentur. Interim autem hinc fiet

$$dv=\frac{e^{\frac{a}{\beta x}}x^{\frac{b}{\beta}}dx}{\alpha-\beta x}.$$

Corollarium 2.

§. 9. Alter casus peculiarem integrationem postulans erit $\beta=0$; tum autem erit $\frac{dQ}{Q}=\frac{dx(a-bx)}{\alpha x}$, vnde fit $lQ=\frac{a}{\alpha}lx-\frac{bx}{\alpha}$, ideoque $Q=x^{\frac{a}{\alpha}}.e^{\frac{-bx}{\alpha}}$, quae formula

casu

casu $x = \infty$ euanescit, si modo fuerit $\frac{b}{\alpha}$ numerus positiuus, sin autem $\frac{b}{\alpha}$ fuerit numerus negatiuus, tum Q euanescit casu $x = -\infty$. Porro vero hoc casu fiet

$$d v = \frac{x^{\frac{a}{\alpha}} . e^{\frac{-b x}{\alpha}} d x}{\alpha - \beta x}$$

Scholion.

§. 8. His in genere obseruatis aliquot casus speciales euoluamus, quibus litteris α, β et a, b certos valores tribuemus, qui ad casus iam satis cognitos perducant

Exemplum I.

§. 9. *Quaerantur formulae integrales, vt fiat*

$$\int x^n d v = \frac{(2n-1)}{2n} \int x^{n-1} d v.$$

Cum igitur hic esse debeat

$$(2n-1) \int x^{n-1} d v = 2n \int x^n d v,$$

erit hoc casu $\alpha = 2$ et $a = -1$, tum vero $\beta = 2$ et $b = 0$; hinc fit

$$\frac{dQ}{Q} = -\frac{dx}{2x(1-x)} = -\frac{dx}{2x} - \frac{dx}{2(1-x)}, \text{ inde integrando}$$

$$lQ = -\tfrac{1}{2} l x + \tfrac{1}{2} l(1-x), \text{ ideoque}$$

$$Q = C \sqrt{\tfrac{1-x}{x}}, \text{ ergo } V = x^n \sqrt{\tfrac{1-x}{x}}.$$

Porro cum hic sit $d v = \frac{Q d x}{2(1-x)}$, erit

$$d v = \frac{d x \sqrt{\frac{1-x}{x}}}{2(1-x)} = \frac{C d x}{2\sqrt{(x - x x)}},$$

sumto ergo $C = 2$ erit $d v = \frac{d x}{\sqrt{(x - x x)}}$, et formula nostra generalis:

$\int x^{n-1}$

$$\int x^{n-1}\,dv = \int \frac{x^{n-1}\,dx}{\sqrt{(x-xx)}},$$

vnde cum fit $V = x^n \sqrt{\frac{1-x}{x}}$, haec quantitas manifesto euanescit sumto $x = 1$, ita vt nostra formula, si post integrationem statuatur $x = 1$, quaesito satisfaciat. Quod si iam ponamus $x = yy$, ista formula induet hanc formam: $2\int \frac{y^{2n-2}\,dy}{\sqrt{(1-yy)}}$, quae, posito post integrationem $y = 1$, praebet hanc relationem:

$$\int \frac{y^{2n}\,dy}{\sqrt{(1-yy)}} = \frac{2n-1}{2n}\int \frac{y^{2n-2}\,dy}{\sqrt{(1-yy)}},$$

quae continet relationes supra § 2 commemoratas; hinc enim fiet

$$\int \frac{yy\,dy}{\sqrt{(1-yy)}} = \frac{1}{2}\int \frac{dy}{\sqrt{(1-yy)}},$$
$$\int \frac{y^4\,dy}{\sqrt{(1-yy)}} = \frac{3}{4}\int \frac{yy\,dy}{\sqrt{(1-yy)}} \text{ et}$$
$$\int \frac{y^6\,dy}{\sqrt{(1-yy)}} = \frac{5}{6}\int \frac{y^4\,dy}{\sqrt{(1-yy)}}.$$

Exemplum 2.

§. 10. *Quaerantur formulae integrales, vt fiat*

$$\int x^n\,dv = \frac{\alpha n - 1}{\alpha n}\int x^{n-1}\,dv.$$

Cum igitur hic esse debeat

$$(\alpha n - 1)\int x^{n-1}\,dv = \alpha n \int x^n\,dv,$$

erit hoc casu $a = -1$, $\beta = \alpha$ et $b = 0$, vnde per formulas supra datas colligitur

$$Q = C\,x^{\frac{-1}{\alpha}}(\alpha - \alpha x)^{\frac{-a}{\alpha}} = C\,x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}$$

quae quantitas manifesto euanescit posito $x = 1$. Tum autem erit

dv

$$dv = \frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}dx}{(1-x)},$$

vnde formula noſtra generalis erit

$$\int x^{n-1}dv = \int x^{n-\frac{1}{\alpha}-1}(1-x)^{+\frac{1}{\alpha}-1}dx = \int \frac{x^{n-\frac{1}{\alpha}-1}dx}{(1-x)^{1-\frac{1}{\alpha}}},$$

quae concinnior redditur, faciendo $x = y^{\alpha}$, tum enim ea induet hanc formam: $\int \frac{y^{\alpha n-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}$, vbi iterum poſt integrationem ſtatui debet $y = 1$. Erit hinc

$$\int \frac{y^{\alpha n+\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha n-1}{\alpha n}\int \frac{y^{\alpha n-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}},$$

atque hinc orientur ſequentes caſus ſpeciales:

$$\int \frac{y^{2\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha-1}{\alpha}\int \frac{y^{\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} \text{ et}$$

$$\int \frac{y^{3\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{2\alpha-1}{2\alpha}\int \frac{y^{2\alpha-2}dy}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}.$$

§. 11. Hinc igitur ſi ſumatur $\alpha = 1$, vt fieri debeat

$$\int x^{n}dv = \frac{n-1}{n}\int x^{n-1}dv,$$

formula noſtra generalis iam in y expreſſa erit $\int y^{n-2}dy$, cuius ergo valor eſt $\frac{1}{n-1}y^{n-1} = \frac{1}{n-1}$, vnde tota ſeries noſtrarum formularum integralium abibit in hanc:

$$\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \text{ etc.}$$

§. 12. Sumamus etiam $\alpha = \frac{1}{2}$, et iam non amplius opus erit ad y procedere. Hoc igitur casu erit

$$Q = \frac{(1-x)^2}{xx} \text{ et } dv = \frac{(1-x)dx}{xx}$$

vnde formula nostra generalis fit

$$\int x^{n-1} dv = \int x^{n-3} (1-x) dx,$$

cuius ergo valor algebraice expressus erit

$$\frac{1}{n-2} x^{n-2} - \frac{1}{n-1} x^{n-1} = \frac{1}{(n-1)(n-2)}$$

vnde series nostrarum formularum euadet

$$\frac{1}{0.-1}, \frac{1}{0.1}, \frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}, \text{ etc.}$$

Exemplum 3.

§. 13. *Quaerantur formulae integrales, vt sit*

$$\int x^n dv = n \int x^{n-1} dv.$$

Cum igitur esse debeat

$$n \int x^{n-1} dv = 1. \int x^n dv, \text{ erit}$$

$$\alpha = 1, \quad a = 0, \quad b = 1, \quad \beta = 0.$$

Cum igitur sit $\beta = 0$, casus Coroll. 2. hic locum habet, indeque erit $Q = e^{-x}$ ideoque $V = e^{-x}. x^n$, quae quantitas his duobus casibus euanescit: $x = 0$ et $x = \infty$. Porro vero erit $dv = e^{-x} dx$, hincque formula nostra generalis fiet $\int x^{n-1} dx. e^{-x}$, vnde ipsi seriei termini ab initio sequenti modo se habebunt:

$$\int e^{-x} dx, \int e^{-x} x dx, \int e^{-x} xx dx, \int e^{-x} x^3 dx \text{ etc.}$$

quibus integratis ita vt euanescant posito $x = 0$, tum vero posito $x = \infty$, orietur sequens series satis simplex:

$$1, 1, 1.2, 1.2.3, 1.2.3.4, 1.2.3.4.5, \text{ etc.}$$

quae

quae est series hypergeometrica *Wallisii*, cuius ergo terminus generalis est

$$\int x^{n-1} e^{-x} dx = 1.2.3.4 \ldots (n-1).$$

§. 14. Ope ergo huius termini generalis hanc seriem interpolare licebit. Ita si quaeratur terminus medius inter duos primos, poni debet $n = \frac{3}{2}$, ac valor huius termini erit $\int e^{-x} dx \sqrt{x}$, cuius autem valor nullo modo algebraice exprimi potest. Inueni autem singulari modo hunc ipsum terminum aequari $\frac{1}{2}\sqrt{\pi}$, denotante π peripheriam circuli cuius diameter $= 1$, vnde hic vicissim cognoscimus esse $\int e^{-x} dx \sqrt{x} = \frac{\sqrt{\pi}}{2}$, posito scilicet post integrationem $x = \infty$. Terminus autem hunc praecedens, indici $\frac{1}{2}$ respondens, erit $= \sqrt{\pi}$, cui ergo aequatur formula $\int \frac{e^{-x} dx}{\sqrt{x}}$. Quod si hic ponamus $e^{x} = y$, ita vt posito $x = 0$ fit $y = 1$, at posito $x = \infty$ fiat $y = \infty$, tum ergo ista formula $\int \frac{e^{-x} dx}{\sqrt{x}}$ abit in hanc $\int \frac{dy}{yy\sqrt{ly}}$, quae formula si ita integretur vt euanescat posito $y = 1$, tum vero fiat $y = \infty$, praebet valorem ipsius $\sqrt{\pi}$. Si porro fiat $y = \frac{1}{z}$, erunt termini integrationis $z = 1$, et $z = 0$, et formula integralis erit

$$-\int \frac{dz}{\sqrt{-lz}} \begin{bmatrix} \text{a} & z = 1 \\ \text{ad} & z = 0 \end{bmatrix} = \sqrt{\pi},$$

siue permutatis terminis integrationis erit

$$\int \frac{dz}{\sqrt{-lz}} \begin{bmatrix} \text{a} & z = 0 \\ \text{ad} & z = 1 \end{bmatrix} = \sqrt{\pi},$$

quemadmodum iam olim obseruaui.

Exemplum 4.

§. 15. *Quaerantur formulae integrales, vt sit*
$\int x^n dv = \frac{1}{n}\int x^{n-1} dv$, *siue*
$\int x^{n-1} dv = n\int x^n dv$.

Hic est $\alpha = 0$ et $a = 1$, $\beta = 1$ et $b = 0$; qui ergo est casus in Coroll. 1. tractatus, vnde colligitur fore $Q = e^{\frac{1}{x}}$; ideoque $V = x^n e^{\frac{1}{x}}$, quae formula nequidem euanescit sumto $x = 0$, quandoquidem formula $e^{\frac{1}{0}}$ aequiualet infinito infinitesimae potestatis. Hic autem miro modo euenit, vt casus $x = -0$ reddat formulam $e^{\frac{-1}{0}}$ subito euanescentem. Scilicet, si ω denotet quantitatem infinite paruam, erit $e^{\frac{1}{\omega}} = \infty^{\infty}$, tum vero repente fiet $e^{\frac{-1}{\omega}} = \frac{1}{\infty^{\infty}} = 0$, quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem $dv = -e^{\frac{1}{x}}\frac{dx}{x}$, ita vt formula nostra generalis futura sit $-\int x^{n-2} dx\, e^{\frac{1}{x}}$, quae autem nobis nullum vsum praestare potest.

§. 16. Quod si hic ponamus $\frac{1}{x} = y$, formula ista generalis transit in hanc: $+\int\frac{e^y dy}{y^n}$. At vero nunc erit $V = \frac{e^y}{y^n}$, quae formula euanescit posito $y = -\infty$. Quomodocunque autem hanc expressionem transformemus, semper idem incommodum occurret. Interim tamen etiam hunc casum sequenti modo resoluere licebit. Sit enim seriei, quam quaerimus

rimus, primus terminus $=\omega$, ex quo per regulam praescriptam sequentes ordine ita procedent

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & & n \\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1.2}, & \frac{\omega}{1.2.3}, & \frac{\omega}{1.2.3.4}, & \ldots\ldots & & \frac{\omega}{1.2.3\ldots(n-1)} \end{array}$$

Supra autem vidimus huius formulae $1.2.3.4.5\ldots(n-1)$ valorem exprimi per hoc integrale: $\int x^{n-1}e^{-x}dx$, integratione ab $x=0$ ad $x=\infty$ extensa; tantum igitur opus est vt hanc formulam integralem in denominatorem transferamus, et seriei quam quaerimus terminus generalis erit

$$\frac{1}{\int x^{n-1}e^{-x}dx},$$

vnde satis intelligitur, negotium non per simplicem formulam integralem expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantitas V non duobus casibus euanescere potest; tum enim tantum opus est fractionem $\frac{\alpha n+a}{\beta n+b}$ inuertere, atque formulam integralem in denominatorem transferre.

Scholion.

§. 17. Nisi sit vel $\alpha=0$ vel $\beta=0$, quos casus iam expediuimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae α et β sunt aequales vnitati. Cum enim esse debeat

$$\int x^n dv=\frac{\alpha n+a}{\beta n+b}\int x^{n-1}dv,$$

ponatur $x=\frac{\alpha y}{\beta}$, fietque

$$\frac{\alpha}{\beta}\int y^n dv=\frac{\alpha n+a}{\beta n+b}\int y^{n-1}dv,$$

quae aequatio reducitur ad hanc formam:

 $\int y^n$

$$\int y^n dv = \frac{n + a:\alpha}{n + b:\beta} \int y^{n-1} dv.$$

Quod si iam nunc loco $\frac{a}{\alpha}$ scribamus a, et b loco $\frac{b}{\beta}$, resoluenda erit haec formula:

$$\int y^n dv = \frac{n+a}{n+b} \int y^{n-1} dv,$$

cuius resolutio, si loco x scribamus y et loco litterarum α et β vnitatem, ex superiori solutione praebet primo

$$Q = C y^a (1 - y)^{b-a},$$

quod ergo euanescit posito $y = 1$, si modo fuerit $b > a$; tum autem erit ipsa formula

$$\int y^{n-1} dv = C \int y^{n+a-1} dy (1 - y)^{b-a-1};$$

sin autem fuerit $b < a$, haec solutio, vti vidimus, locum habere nequit; verum hoc casu pro termino nostrae seriei assumi debet haec forma: $\frac{1}{\int y^{n-1} dv}$, ita vt tum esse debeat

$$\frac{1}{\int y^n dv} = \frac{n+a}{n+b} \cdot \frac{1}{\int y^{n-1} dv}, \text{ siue}$$

$$\int y^n dv = \frac{n+b}{n+a} \int y^{n-1} dv,$$

cuius resolutio permutatis litteris a et b praebet

$$Q = C y^b (1 - y)^{a-b}$$

quae pariter casu $y = 1$ euanescit, si fuerit $a > b$, atque tum erit formula generalis

$$\int y^{n-1} dv = C \int y^{n+b-1} dy (1 - y)^{a-b-1}.$$

Siue igitur sit $b > a$ siue $a > b$, solutio nulla amplius laborat difficultate.

§. 18. Sin autem fuerit vel $\alpha = 0$ vel $\beta = 0$, loco alterius etiam scribi poterit vnitas; vnde si esse debeat

$\int x^n$

$$\int x^n\, dv = \frac{n+a}{b} \int x^{n-1}\, dv,$$

ob $\alpha = 1$ et $\beta = 0$ ſolutio noſtra generalis dat

$$\frac{dQ}{Q} = \frac{dx}{x}(a - bx)$$

vnde colligitur $Q = C x^a . e^{-bx}$, quae formula euaneſcit poſito $x = \infty$, ſi modo b fuerit numerus poſitiuus; tum autem fit terminus generalis

$$\int x^{n-1}\, dv = C \int x^{n+a-1}\, dx . e^{-x}.$$

At vero numerus b negatiuus eſſe nequit, quia alioquin conditio praeſcripta eſſet incongrua.

§. 19. Conſideremus etiam alterum caſum, quo $\alpha = 0$ et $\beta = 1$, ideoque conditio praeſcripta

$$\int x^n\, dv = \frac{a}{n+b} \int x^{n-1}\, dv,$$

vnde fit

$$\frac{dQ}{Q} = -\frac{dx}{xx}(a - bx).$$

Hinc autem pro Q orietur valor, qui praeter caſum $x = 0$ euaneſcere non poſſet; quam ob cauſſam formula generalis ſtatui debet $\frac{1}{\int x^{n-1}\, dv}$, ita vt eſſe debeat

$$\int x^n\, dv = \frac{n+b}{a} \int x^{n-1}\, dv$$

vnde prodit

$$\frac{dQ}{Q} = \frac{dx}{x}(b - ax), \text{ ideoque } Q = C e^{-ax} . x^b,$$

quae expreſſio euaneſcit poſito $x = \infty$, quoniam a neceſſario debet eſſe numerus poſitiuus; tum autem erit

$$dv = C e^{-ax} . x^b\, dx,$$

vnde

vnde formula generalis seriei erit

$$\frac{1}{C\int x^{a+b-1}\,dx.\,e^{-ax}}$$

Problema II.

§. 20. *Denotet* T *terminum indici* n *respondentem in serie quam considerandam suscepimus, at vero* T′ *terminum sequentem, atque proponatur haec conditio adimplenda:*

$$T' = \frac{(\alpha n + a)(\alpha' n + a')}{(\beta n + b)(\beta' n + b')}\,T$$

Solutio.

Quoniam hic valores geminati occurrunt, huic conditioni commodissime satisfiet, si terminus generalis T tanquam productum ex duobus factoribus spectetur. Statuatur igitur $T = RS$, sitque terminus sequens $= R'S'$, et quaerantur formulae R et S, vt fiat

$$R' = \frac{\alpha n + a}{\beta n + b}\,R \text{ et } S' = \frac{\alpha' n + a'}{\beta' n + b'}\,S,$$

tum enim sumendo $T = RS$ conditioni praescriptae manifesto satisfiet Hoc igitur modo pro R et S vel huiusmodi formulae: $\int x^{n-1}dv$, vel inuersae $\frac{1}{\int x^{n-1}dv}$ reperientur, id quod pro solutione generali sufficit, vnde rem exemplo illustremus.

Exemplum.

§. 21. *Quaeratur formula generalis* T, *vt fiat*

$$T' = \frac{nn - cc}{nn}\,T.$$

Resoluamus igitur T in duos factores R et S, ac statuamus

$$R' = \frac{n-c}{n}\,R \text{ et } S' = \frac{n+c}{n}\,S.$$

Pro

Pro priore forma si statuamus $R = \int x^{n-1} dv$, ex solutione generali, vbi erit $\alpha = 1$, $a = -c$, $\beta = 1$ et $b = 0$, fiet

$$Q = C x^{-c} (1-x)^c,$$

quae forma manifesto euanescit posito $x = 1$, hincque quia fit

$$V = C x^{n-c} (1-x)^c,$$

haec forma etiam casu $x = 0$ euanescit, si modo n fuerit $> c$, id quod tuto assumi potest, quia exponentem n successiue in infinitum crescere assumimus, ac plerumque pro c fractiones tantum accipi solent. Hinc ergo erit

$$R = C \int x^{n-c-1} (1-x)^{c-1} dx.$$

§. 22. Hinc iam alter valor litterae S deduci posset, scribendo tantum $-c$ loco c, tum autem non amplius fieret $Q = 0$ posito $x = 1$, quamobrem pro S formulam inuersam $\frac{1}{\int x^{n-1} dv}$ assumi oportet, vt esse debeat

$$\int x^n dv = \frac{n}{n+c} \int x^{n-1} dv,$$

vbi cum sit $\alpha = 1$, $a = 0$, $\beta = 1$ et $b = c$, reperitur $Q = C(1-x)^c$, quae forma manifesto fit $= 0$ posito $x = 1$, hinc autem prodit

$$dv = C(1-x)^{c-1} dx,$$

ergo habebimus

$$S = \frac{1}{C \int x^{n-1} (1-x)^{c-1} dx},$$

consequenter formula nostra generalis quaesita erit

$$T = \frac{\int x^{n-c-1} (1-x)^{c-1} dx}{\int x^{n-1} (1-x)^{c-1} dx}.$$

§. 23. Quod si ergo nostrae seriei per factores procedentis primum terminum ponamus $= A$, ipsa series erit

$$\text{I. } A. \quad \text{II. } \frac{1-cc}{1}. A, \quad \text{III. } \frac{1-cc}{1}. \frac{4-cc}{4}. A, \quad \text{IV. } \frac{1-cc}{1}. \frac{4-cc}{4}. \frac{9-cc}{9} A, \text{ etc.}$$

vnde si sumamus $c = \frac{1}{2}$, erit haec series

$$A, \frac{1.3}{2.2}. A, \frac{1.3}{2.2}. \frac{3.5}{4.4}. A, \frac{1.3}{2.2}. \frac{3.5}{4.4}. \frac{5.7}{6.6}. A, \text{ etc.}$$

cuius ergo terminus indici n respondens est

$$\frac{\int x^{n-\frac{3}{2}}(1-x)^{-\frac{1}{2}}dx}{\int x^{n-1}(1-x)^{-\frac{1}{2}}dx},$$

qui posito $x = yy$ transit in hanc formam:

$$\frac{\int y^{2n-2}(1-yy)^{-\frac{1}{2}}dy}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}dy},$$

vnde patet, terminum primum fore

$$A = \int \frac{dy}{\sqrt{(1-yy)}} : \int \frac{ydy}{\sqrt{(1-yy)}} = \frac{\pi}{2}$$

posito scilicet post integrationem $y = 1$.

Problema.

Denotet T *terminum seriei indici* n *respondentem, sintque* T′ *et* T″ *termini sequentes pro indicibus* n + 1 *et* n + 2, *si proponatur inter ternos terminos se insequentes talis relatio, vt sit*

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T'';$$

inuestigare formulam pro T, *qua terminus generalis huius seriei exprimatur.*

Solutio.

Solutio.

§. 24. Assumatur pro T formula integralis $\int x^{n-1} dv$, huiusque integrale ita capiatur, vt euanescat posito $x=0$, eruntque termini sequentes

$$T' = \int x^n dv \text{ et } T'' = \int x^{n+1} dv,$$

siquidem post integrationem variabili x certus valor determinatus tribuatur. Quamdiu autem haec quantitas x vt variabilis spectatur, ponamus esse

$$(\alpha n+a)T = (\beta n+b)T' + (\gamma n+c)T'' + x^n Q,$$

ac perspicuum est Q eiusmodi functionem esse debere ipsius x, quae euanescat, si loco x valor ille determinatus substituatur, quem autem a ciphra diuersum esse oportet, quoniam iam assumsimus, omnes istas formulas in nihilum abire posito $x=0$. Quodsi vero, absoluto calculo, huic conditioni nullo modo satisfieri poterit, id erit indicio, problema nostrum hac ratione resolui non posse, vt scilicet eius terminus generalis T per talem formulam differentialem simplicem $\int x^{n-1} dv$ exhibeatur.

§. 25. Differentiemus nunc aequationem modo stabilitam, ac diuisione facta per x^{n-1} sequens prodibit aequatio:

$$(\alpha n+a)dv = (\beta n+b)x\,dv + (\gamma n+c)xx\,dv + nQ\,dx + x\,dQ,$$

quae, quia termini littera n affecti seorsim se destruere debent, discerpetur in binas sequentes aequationes:

$$1^\circ\ \alpha\,dv = \beta\,x\,dv + \gamma\,x\,x\,dv + Q\,dx,$$
$$2^\circ\ a\,dv = b\,x\,dv + c\,x\,x\,dv + x\,dQ,$$

 ex

ex quarum priore fit

$$dv = \frac{Q\,dx}{\alpha - \beta x - \gamma xx},$$

ex altera vero fit

$$dv = \frac{x\,dQ}{a - bx - cxx};$$

quorum valorum posterior per priorem diuisus praebet

$$\frac{dQ}{Q} = \frac{dx\,(a - bx - cxx)}{x\,(\alpha - \beta x - \gamma xx)},$$

ex cuius ergo integratione valor ipsius Q elici debet, quo facto facile patebit, vtrum is certo quodam casu praeter $x = 0$ euanescere possit. Imprimis autem hic notari conuenit, si hoc integrale inuoluat huiusmodi factorem $e^{\frac{\lambda}{x}}$; tum solutionem quoque successu esse carituram, quandoquidem posito $x = 0$ iste factor tantam inuoluet infiniti potestatem, vt, etiamsi per x^n multiplicetur, productum etiamnum infinitum maneat.

§. 26. Quod si igitur his conditionibus praescriptis satisfacere licuerit, tum inuento valore litterae Q, quem ponamus fieri $= 0$ posito $x = f$, habebitur

$$dv = \frac{Q\,dx}{\alpha - \beta x - \gamma xx},$$

et formula generalis naturam seriei complectens erit

$$T = \int x^{n-1}\,dv = \int \frac{x^{n-1}\,Q\,dx}{\alpha - \beta x - \gamma xx},$$

quippe cuius integrale, a termino $x = 0$ vsque ad terminum $x = f$ extensum, praebebit valorem termini T, indici cuicunque n respondentis.

Scholion.

Scholion.

§. 27. Inuenta autem tali relatione inter ternos terminos cuiuspiam ſeriei ſibi inuicem ſuccedentes, inde more ſolito formari poterit fractio continua, cuius valorem aſſignare licebit. Si enim characteres T', T'', T''', T^{IV}, etc. denotent ordine omnes terminos poſt T ſequentes in infinitum, ex relationibus, quas inter ſe tenent, ſequentes formulae deducentur. Ex relatione

$$(\alpha n + a)T = (\beta n + b)T' + (\gamma n + c)T''$$

deducitur

$$(\alpha n + a)\frac{T}{T'} = \beta n + b + \frac{(\gamma n + c)(\alpha n + \alpha + a)}{(\alpha n + \alpha + a)T' : T''}.$$

Ex relatione ſequente

$$(\alpha n + \alpha + a)T' = (\beta n + \beta + b)T'' + (\gamma n + \gamma + c)T'''$$

deducitur

$$(\alpha n + \alpha + a)\frac{T'}{T''} = \beta n + \beta + b + \frac{(\gamma n + \gamma + c)(\alpha n + 2\alpha + a)}{(\alpha n + 2\alpha + a)T'' : T'''}.$$

Simili modo ſequentes relationes ſuppeditabunt:

$$(\alpha n + 2\alpha + a)\frac{T''}{T'''} = \beta n + 2\beta + b + \frac{(\gamma n + 2\gamma + c)(\alpha n + 3\alpha + a)}{(\alpha n + 3\alpha + a)T''' : T''''};$$

$$(\alpha n + 3\alpha + a)\frac{T'''}{T''''} = (\beta n + 3\beta + b) + \frac{(\gamma n + 3\gamma + c)(\alpha n + 4\alpha + a)}{(\alpha n + 4\alpha + a)T'''' : T'''''};$$

vnde manifeſtum eſt, ſi in prima formula continuo ſequentes valores ordine ſubſtituantur, prodituram eſſe fractionem continuam, cuius valor aequalis erit formulae $(\alpha n + a)\frac{T}{T'}$.

§. 28. Quod ſi ergo loco n ſucceſſiue ſcribamus numeros 1, 2, 3, 4, etc., ſequens problema circa fractiones continuas reſoluere poterimus.

Problema.

Proposita fractione continua huius formae:

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{etc.}}}}}}$$

eius valorem inuestigare.

Solutio.

Consideretur in genere ista relatio inter ternas quantitates sibi succedentes T, T′, T″, quae sit

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T'',$$

atque ex praecedente Problemate quaeratur valor ipsius T; siquidem fieri potest, hoc modo expressus:

$$T=\int x^{n-1}\,dv=\int\frac{x^{n-1}Q\,dx}{\alpha-\beta x-\gamma xx},$$

cuius integrale ab $x=0$ vsque ad $x=f$ extendatur, quae formula inuenta ponatur

$$\int\frac{Q\,dx}{\alpha-\beta x-\gamma xx}=A \text{ et } \int\frac{xQ\,dx}{\alpha-\beta x-\gamma xx}=B,$$

ita vt A et B sint valores ipsius T, pro casibus $n=1$ et $n=2$, quibus definitis fractionis continuae propositae valor per praecedentia erit $=\frac{(\alpha+a)A}{B}$. Hanc igitur inuestigationem ad sequentia exempla accommodemus.

Exem-

Exemplum 1.

§. 29. *Inuestigare valorem fractionis continuae notissimae, quam olim* Brouncherus *pro quadratura circuli protulit, quae est*

$$2+\cfrac{1.\ 1}{2+\cfrac{3.\ 3}{2+\cfrac{5.\ 5}{2+\text{etc.}}}}$$

Quia omnes partes integrae laeuam respicientes sunt constantes $=2$, pro nostra forma generali fiet

$$\beta+b=2,\quad 2\beta+b=2,\quad 3\beta+b=2 \text{ etc.}$$

erit ergo $\beta=0$ et $b=2$; at pro numeratoribus sequentium fractionum, quandoquidem constant binis factoribus, erit pro factoribus prioribus

$$\gamma+c=1,\quad 2\gamma+c=3,\quad 3\gamma+c=5,\quad 4\gamma+c=7,$$

vnde concluditur $\gamma=2$ et $c=-1$; pro alteris vero erit

$$2\alpha+a=1,\quad 3\alpha+a=3,\quad 4\alpha+a=5 \text{ etc.}$$

vnde $\alpha=2$ et $a=-3$. Ex his autem valoribus colligimus hanc aequationem

$$\frac{dQ}{Q}=-\frac{dx(3+2x-xx)}{2x(1-xx)},$$

quae per $1+x$ depressa praebet

$$\frac{dQ}{Q}=-\frac{dx(3-x)}{2x(1-x)},$$

vnde integrando fit

$$lQ=-\tfrac{3}{2}lx+l(1-x) \text{ et hinc } Q=\frac{1-x}{x^{\frac{3}{2}}},$$

ex

ex quo valore porro sequitur

$$A = \int \frac{(1-x)\,dx}{2x^{\frac{3}{2}}(1-xx)} = \int \frac{dx}{2x(1+x)\sqrt{x}}$$

$$B = \int \frac{(1-x)\,dx}{2x^{\frac{1}{2}}(1-xx)} = \int \frac{dx}{2(1+x)\sqrt{x}}.$$

§. 30. In his autem valoribus istud incommodum deprehenditur, quod prius integrale euanescens reddi nequit posito $x = 0$. Hoc autem incommodum facile remoueri potest, si fractionem continuam supremo membro truncemus et quaeramus valorem istius fractionis:

$$2 + \cfrac{3.\,3}{2 + \cfrac{5.\,5}{2 + \text{etc.}}}$$

qui si repertus fuerit $= s$, erit ipsius propositae valor $= b + \frac{1}{s}$. Nunc vero, comparatione instituta, fit quidem vt ante $\beta = 0$ et $b = 2$, tum vero $\gamma = 2$ et $c = +1$, $\alpha = 2$ et $a = -1$, vnde sequitur

$$\frac{dQ}{Q} = -\frac{dx(1+2x+xx)}{2x(1-xx)} = -\frac{dx(1+x)}{2x(1-x)},$$

vnde integrando fit

$$lQ = -\tfrac{1}{2}lx + l(1-x) \text{ ideoque } Q = \frac{1-x}{\sqrt{x}},$$

ex quo valore iam habebimus

$$A = \int \frac{(1-x)\,dx}{2(1-xx)\sqrt{x}} = \tfrac{1}{2}\int \frac{dx}{(1+x)\sqrt{x}} \text{ et}$$

$$B = \tfrac{1}{2}\int \frac{dx\sqrt{x}}{1+x}$$

vbi cum sit $Q = \frac{1-x}{\sqrt{x}}$, eius valor manifesto euanescit posito $x = 1$, quamobrem illa integralia a termino $x = 0$, vsque ad $x = 1$ sunt extendenda.

§. 31.

§. 31. Quo nunc haec integralia facilius eruamus, ſtatuamus $x = zz$, ita vt termini integrationis etiamnunc ſint $z = 0$ et $z = 1$, eritque

$$A = \int \frac{dz}{1+zz} = A\ \text{tang.}\ z = \frac{\pi}{4} \text{ et}$$

$$B = \int \frac{zz\,dz}{1+zz} = 1 - \frac{\pi}{4},$$

ſicque habebimus $s = \frac{\pi}{4-\pi}$, quocirca ipſius fractionis *Brouncherianae* valor eſt $1 + \frac{4}{\pi}$, omnino vti olim *Brouncherus* iam inuenerat.

Exemplum 2.

§. 31. *Inueſtigare valorem huius fractionis continuae* Brouncheriana *latius patentis:*

$$b + \cfrac{1.\ 1}{b + \cfrac{3.\ 3}{b + \cfrac{5.\ 5}{b + \text{etc.}}}}$$

Vt hic incommodum ſuperius euitemus, omittamus membrum ſupremum et quaeramus

$$s = b + \cfrac{3.\ 3}{b + \cfrac{5.\ 5}{b + \text{etc.}}}$$

quandoquidem tum erit valor quaeſitus $= b + \frac{1}{s}$. Nunc igitur erit $\beta = 0$ et $b = b$, $\gamma = 2$, $c = 1$, $\alpha = 2$ et $a = -1$, vnde fit

$$\frac{dQ}{Q} = -\frac{dx(1+bx+xx)}{2x(1-xx)}, \text{ ac proinde}$$

$$lQ = -\frac{1}{2}lx - \frac{b-2}{4}l(1+x) + \frac{b+2}{4}l(1-x),$$

 Q =

hincque

$$Q = \frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}} \sqrt{x}},$$

quae formula manifesto fit $= 0$ ponendo $x = 1$, siquidem $b + 2$ fuerit numerus positiuus, vnde fit

$$dv = \frac{(1-x)^{\frac{b-2}{4}}}{2(1+x)^{\frac{b+2}{4}} \sqrt{x}}.$$

Hinc autem colligetur

$$A = \tfrac{1}{2} \int \frac{(1-x)^{\frac{b-2}{4}}\, dx}{(1+x)^{\frac{b+2}{4}} \sqrt{x}} \text{ et}$$

$$B = \tfrac{1}{2} \int \frac{(1-x)^{\frac{b-2}{4}}\, dx \sqrt{x}}{(1+x)^{\frac{b+2}{4}}},$$

siue ponendo $x = zz$ habebimus

$$A = \int \frac{(1-zz)^{\frac{b-2}{4}}\, dz}{(1+zz)^{\frac{b+2}{4}}} \text{ et}$$

$$B = \int \frac{(1-zz)^{\frac{b-2}{4}}\, zz\, dz}{(1+zz)^{\frac{b+2}{4}}},$$

quae ambo integralia a $z = 0$ vsque ad $z = 1$ sunt extendenda. Ex his autem valoribus A et B erit $s = \frac{A}{B}$; ipsius igitur fractionis propositae valor erit $= b + \frac{1}{s} = b + \frac{B}{A}$.

§. 42.

§. 32. Quod si hic ponatur $b = 2$, prodit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes $\frac{b-2}{4}$ et $\frac{b+2}{4}$ non sunt numeri integri, tum litteras A et B neque per arcus circulares, neque per logarithmos exprimere licet. Veluti si fuerit $b = 4$, erit

$$A = \int \frac{dz\sqrt{(1-zz)}}{(1+zz)^{\frac{3}{2}}},$$

cuius valor per arcus ellipticos exhiberi posset. At si b fuerit numerus impar, hi valores multo magis euadunt transcendentes, ita vt his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi fiant numeri integri, totum negotium per arcus circulares expedire licebit.

§. 33. Exponentes autem illi $\frac{b-2}{4}$ et $\frac{b+2}{4}$ erunt numeri integri, quoties fuerit b numerus huius formae:

$$b = 4i + 2$$

tum enim erit

$$A = \int \frac{(1-zz)^i dz}{(1+zz)^{i+1}} \text{ et}$$

$$B = \int \frac{(1-zz)^i zz dz}{(1+zz)^{i+1}};$$

quos ergo casus quomodo euolui oporteat operae pretium erit docere, quoniam *Wallisius* eos iam est contemplatus.

§. 34. Quoniam hoc negotium totum redit ad reductionem huiusmodi formularum integralium ad formas

 sim-

simpliciores, consideremus in genere formam $P = \frac{z^m}{(1+zz)^n}$, cuius differentiale sub sequentibus formis exhiberi potest:

$$1^\circ)\quad dP = \frac{m z^{m-1} dz}{(1+zz)^n} - \frac{2n z^{m+1} dz}{(1+zz)^{n+1}}$$

$$2^\circ)\quad dP = \frac{m z^{m-1} dz}{(1+zz)^{n+1}} - \frac{(2n-m) z^{m+1} dz}{(1+zz)^{n+1}}$$

$$3^\circ)\quad dP = -\frac{(2n-m) z^{m-1} dz}{(1+zz)^n} + \frac{2n z^{m-1} dz}{(1+zz)^{n+1}}$$

vnde hanc triplicem reductionem integralium deducimus:

$$\text{I.}\quad \int \frac{z^{m+1} dz}{(1+zz)^{n+1}} = \frac{m}{2n} \int \frac{z^{m-1} dz}{(1+zz)^n} - \frac{1}{2n} \cdot \frac{z^m}{(1+zz)^n}$$

$$\text{II.}\quad \int \frac{z^{m+1} dz}{(1+zz)^{n+1}} = \frac{m}{2n-m} \int \frac{z^{m-1} dz}{(1+zz)^{n+1}} - \frac{1}{2n-m} \cdot \frac{z^m}{(1+zz)^n}$$

$$\text{III.}\quad \int \frac{z^{m-1} dz}{(1+zz)^{n+1}} = \frac{2n-m}{2n} \int \frac{z^{m-1} dz}{(1+zz)^n} + \frac{1}{2n} \cdot \frac{z^m}{(1+zz)^n}$$

quarum reductionum ope casibus $b = 4i + 2$ totum negotium absolui et ad formulam $\frac{\pi}{4}$ reduci poterit, siquidem post integrationem sumatur $z = 1$.

§. 35. Sit $i = 1$ ideoque $b = 6$ eritque
$A = \int \frac{(1-zz)dz}{(1+zz)^2}$ et $B = \int \frac{(1-zz)zz\,dz}{(1+zz)^2}$.

Nunc igitur reperiemus per reductionem tertiam

$$\int \frac{dz}{(1+zz)^2} = \frac{1}{2} \int \frac{dz}{1+zz} + \frac{1}{2} \cdot \frac{z}{1+zz} = \frac{\pi}{8} + \frac{1}{4}$$

et per reductionem primam

$$\int \frac{zz\,dz}{(1+zz)^2} = \frac{1}{2} \int \frac{dz}{1+zz} - \frac{1}{2} \cdot \frac{z}{1+zz} = \frac{\pi}{8} - \frac{1}{4},$$ porro

$$\int \frac{z^4\,dz}{(1+zz)^2} = \frac{3}{2} \int \frac{zz\,dz}{1+zz} - \frac{1}{2} \cdot \frac{z^3}{1+zz} = \frac{5}{4} - \frac{3\pi}{8}.$$

Ex

Ex his iam valoribus colligitur $A = \frac{1}{2}$ et $B = \frac{\pi}{2} - \frac{3}{2}$, ideoque $\frac{B}{A} = \pi - 3$, quocirca orietur iſta ſummatio:

$$3 + \pi = 6 + \cfrac{1.1}{6 + \cfrac{3.3}{6 + \cfrac{5.5}{6 + \cfrac{7.7}{6 + \text{etc.}}}}}$$

§. 36. Sit nunc $i = 2$ et $b = 10$, eritque

$$A = \int \frac{(1 - zz)^2\, dz}{(1 + zz)^3} \text{ et } B = \int \frac{zz(1 - zz)^2\, dz}{(1 + zz)^3}.$$

Quo harum integralium valores inueſtigemus, ſequentes euoluamus formulas:

$$\int \frac{dz}{(1+zz)^3} = \frac{3}{4}\int \frac{dz}{(1+zz)^2} + \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{3\pi}{32} + \frac{1}{4}$$

$$\int \frac{zz\, dz}{(1+zz)^3} = \frac{1}{4}\int \frac{dz}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{\pi}{32}$$

$$\int \frac{z^4\, dz}{(1+zz)^3} = \frac{3}{4}\int \frac{zz\, dz}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^3}{(1+zz)^2} = \frac{3\pi}{32} - \frac{1}{4}$$

$$\int \frac{z^6\, dz}{(1+zz)^3} = \frac{5}{4}\int \frac{z^4\, dz}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^5}{(1+zz)^2} = \frac{3}{2} - \frac{15\pi}{32}.$$

Ex quibus iam valoribus deducitur $A = \frac{\pi}{8}$ et $B = 2 - \frac{5\pi}{8}$, ideoque $\frac{B}{A} = \frac{16 - 5\pi}{\pi}$, vnde emergit ſequens ſummatio:

$$\frac{5\pi + 16}{\pi} = 10 + \cfrac{1.1}{10 + \cfrac{3.3}{10 + \cfrac{5.5}{10 + \text{etc.}}}}$$

§. 37. Si b eſſet numerus negatiuus, inueſtigatio nulla prorſus laboraret difficultate. Si enim in genere fuerit

$$s = -a + \cfrac{\alpha}{-b + \cfrac{\beta}{-c + \cfrac{\gamma}{-d + \cfrac{\delta}{-e + \text{ etc.}}}}}$$

ſemper erit

$$s = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \text{ etc.}}}}}$$

vnde ſi habeatur valor iſtius expreſſionis, idem negatiue ſumtus dabit valorem illius.

Exemplum 3.

§. 38. *Propoſita ſit fractio continua, cuius valorem inueſtigari oporteat, iſta:*

$$1 + \cfrac{1 \cdot 1}{3 + \cfrac{3 \cdot 3}{5 + \cfrac{5 \cdot 5}{7 + \cfrac{7 \cdot 7}{9 + \text{ etc.}}}}}$$

Quo fractiones ſupra allegatas, omiſſo membro ſupremo, ſint

$$3 + \cfrac{3 \cdot 3}{5 + \cfrac{5 \cdot 5}{7 + \cfrac{7 \cdot 7}{9 + \text{ etc.}}}}$$

eritque $\beta + b = 3$, $2\beta + b = 5$, ideoque $\beta = 2$ et $b = 1$; tum vero vt ante $\alpha = 2$, $a = -1$, $\gamma = 2$ et $c = +1$;

Inuento

inuento autem s erit valor quaesitus $= 1 + \frac{1}{s}$. Nunc igitur habebimus

$$\frac{dQ}{Q} = -\frac{dx(1+x+xx)}{2x(1-x-xx)}. \text{ Est vero}$$

$$\frac{1+x+xx}{x(1-x-xx)} = \frac{1}{x} + \frac{2+2x}{1-x-xx}, \text{ vnde fit}$$

$$lQ = -\tfrac{1}{2}lx - \int\frac{dx(1+x)}{1-x-xx}.$$

Porro vero pro formula $\int\frac{dx(1+x)}{1-x-xx}$ inuenienda, statuamus denominatorem

$$1-x-xx = (1-fx)(1-gx)$$

eritque $f+g=1$ et $fg=-1$, vnde fit

$$f = \frac{1+\sqrt{5}}{2} \text{ et } g = \frac{1-\sqrt{5}}{2}.$$

Nunc statuatur

$$\frac{1+x}{1-x-xx} = \frac{\mathfrak{A}}{1-fx} + \frac{\mathfrak{B}}{1-gx},$$

vnde reperietur

$$\mathfrak{A} = \frac{1+f}{f-g} \text{ et } \mathfrak{B} = -\frac{(1+g)}{f-g}$$

siue substitutis pro f et g valoribus supra datis erit

$$\mathfrak{A} = \frac{\sqrt{5}+3}{2\sqrt{5}} \text{ et } \mathfrak{B} = \frac{\sqrt{5}-3}{2\sqrt{5}},$$

quibus inuentis erit

$$\int\frac{dx(1+x)}{1-x-xx} = -\frac{\mathfrak{A}}{f}l(1-fx) - \frac{\mathfrak{B}}{g}l(1-gx) =$$
$$-\frac{(1+\sqrt{5})}{2\sqrt{5}}l(1-fx) - \frac{(\sqrt{5}-1)}{2\sqrt{5}}l(1-gx)$$

quocirca fiet

$$lQ = -\tfrac{1}{2}lx + \frac{(\sqrt{5}+1)}{2\sqrt{5}}l(1-fx) + \frac{(\sqrt{5}-1)}{2\sqrt{5}}l(1-gx)$$

consequenter

$$Q = \frac{(1-fx)^{\frac{\sqrt{5}+1}{2\sqrt{5}}}(1-gx)^{\frac{\sqrt{5}-1}{2\sqrt{5}}}}{\sqrt{x}},$$

qui

qui valor duobus casibus euanescit: altero quo

$$x = \frac{1}{f} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2},$$

altero vero quo $x = \frac{1}{g} = -\frac{1-\sqrt{5}}{2}$; vtrouis autem vtamur, res eodem redibit.

§. 39. Ex hoc autem valore habebimus

$$A = \int \frac{Q\,dx}{1-x-xx} \text{ et } B = \int \frac{Q\,x\,dx}{1-x-xx},$$

vnde porro deducitur

$$s = (\alpha + a)\frac{A}{B} = \frac{A}{B}$$

hinc propositae fractionis summa erit $1 + \frac{B}{A}$. Hinc autem nihil vlterius concludere licet, ob formulas differentiales non solum irrationales, sed etiam vere transscendentes ob exponentes surdos.

Exemplum 4.

§. 40. *Proposita sit haec fractio continua:*

$$b + \cfrac{1.1}{b + \cfrac{2.2}{b + \cfrac{3.3}{b + \cfrac{4.4}{b + \text{etc.}}}}}$$

vbi est $\beta = 0$, $b = b$.

Nunc consideremus hanc formam:

$$s = b + \cfrac{2.2}{b + \cfrac{3.3}{b + \text{etc.}}},$$

quippe quo valore inuento quaesitus erit $= b + \frac{1}{s}$. Habebimus igitur $\gamma + c = 2$, $2\gamma + c = 3$, ideoque $\gamma = 1$ et $c = 1$,

$e=1$, deinde erit $\alpha=\gamma=1$, $a=0$ et $c=1$. Hinc igitur colligimus

$\frac{dQ}{Q}=-\frac{dx(bx+xx)}{x(1-xx)}=-\frac{dx(b+x)}{1-xx}$, ideoque

$lQ=-\frac{b}{2}l\frac{1+x}{1-x}+\frac{1}{2}l(1-xx)$ hincque

$$Q=\frac{(1-x)^{\frac{b}{2}}\sqrt{(1-xx)}}{(1+x)^{\frac{b}{2}}}=\frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

quae quantitas manifesto euanescit posito $x=1$. Hinc igitur fiet

$$A=\int\frac{Q\,dx}{1-xx}=\int\frac{(1-x)^{\frac{b+1}{2}}dx}{(1+x)^{\frac{b-1}{2}}(1-xx)}=\int\frac{(1-x)^{\frac{b-1}{2}}dx}{(1+x)^{\frac{b+1}{2}}}\text{ et}$$

$$B=\int\frac{x(1-x)^{\frac{b-1}{2}}dx}{(1+x)^{\frac{b+1}{2}}},$$

tum autem erit $s=(\alpha+a)\frac{A}{B}=\frac{A}{B}$ ideoque summa quaesita $b+\frac{B}{A}$.

§. 41. Percurramus nunc casus praecipuos: ac primo sit $b=1$ eritque

$A=\int\frac{dx}{1+x}=l(1+x)=l2$ et

$B=\int\frac{x\,dx}{1+x}=x-\int\frac{dx}{1+x}=1-l2$,

ideoque $b+\frac{B}{A}=\frac{1}{l2}$; ergo hinc prodiit ista summatio:

$$\frac{1}{l\,2} = 1 + \cfrac{1.\,1}{1 + \cfrac{2.\,2}{1 + \cfrac{3.\,3}{1 + \text{etc.}}}}$$

§. 42. Sit nunc $b = 2$ eritque

$$A = \int \frac{dx\,\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}} \text{ et } B = \int \frac{x\,dx\,\sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

Ad has formulas rationales reddendas ſtatuamus

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = z, \text{ eritque } x = \frac{1-zz}{1+zz},$$

vnde terminis integrationis $x = 0$ et $x = 1$ reſpondebunt $z = 1$ et $z = 0$; tum vero erit

$$1 + x = \frac{2}{1+zz} \text{ et } dx = -\frac{4\,z\,dz}{(1+zz)^2},$$

hincque colligitur

$$A = -2\int \frac{zz\,dz}{1+zz} = -2z + 2\,A\,\text{tang.}\,z + 2 - \frac{\pi}{2} = 2 - \frac{\pi}{2},$$

porro fit

$$B = -2\int \frac{zz\,dz}{(1+zz)^2} + 2\int \frac{z^4\,dz}{(1+zz)^2}.$$

Per reductiones igitur ſupra § 35 monſtratas, ſi hic ſcilicet terminos integrationis $z = 1$ et $z = 0$ permutemus, vt habeamus

$$B = +2\int \frac{zz\,dz}{(1+zz)^2} - 2\int \frac{z^4\,dz}{(1+zz)^2}, \text{ erit}$$

$$B = 2\left(\frac{\pi}{8} - \frac{1}{4}\right) - 2\left(\frac{5}{4} - \frac{\pi}{4}\right) = \pi - 3,$$

vnde ſequitur iſta ſummatio:

$\frac{1}{4-\pi}$

$$\frac{2}{4-\pi}=2+\cfrac{1.1}{2+\cfrac{3.3}{2+\cfrac{4.4}{2+\text{etc.}}}}$$

quae *Brounckerianae* simplicitate nihil cedit.

§. 43. Si ponamus $b=0$, fractio continua abit in sequens continuum productum:

$$\frac{1.1}{2.2}\cdot\frac{3.3}{4.4}\cdot\frac{5.5}{6.6}\cdot\frac{7.7}{8.8}\cdot\text{ etc.}$$

hoc autem casu fit

$$A=\int\frac{dx}{\sqrt{(1-xx)}}=\frac{\pi}{2}\text{ et }B=\int\frac{x\,dx}{\sqrt{(1-xx)}}=1$$

vnde istius producti valor colligitur $\frac{2}{\pi}$, id quod egregie convenit cum iam dudum cognitis, quandoquidem hoc productum est ipsa progressio *Wallisiana*.

Exemplum 5.

§. 44. *Proposita sit haec fractio continua, vbi* $\beta=0$, $b=b$ *et numeratores numeri trigonales:*

$$b+\cfrac{1}{b+\cfrac{3}{b+\cfrac{6}{b+\cfrac{10}{b+\text{etc.}}}}}$$

Omisso supremo membro statuamus

$$s=b+\cfrac{3}{b+\cfrac{6}{b+\text{etc.}}}$$

et primo numeratores per producta repraesentemus, hoc modo:

$$3 = 2 \cdot \tfrac{3}{2}, \quad 6 = 3 \cdot \tfrac{4}{2}, \quad 10 = 4 \cdot \tfrac{5}{2} \; .$$

quorum priores comparentur cum formulis

$$\gamma + c, \quad 2\gamma + c, \quad 3\gamma + c,$$

posteriores vero cum formulis $2\alpha + a$, $3\alpha + a$, $4\alpha + a$, eritque $\gamma = 1$, $c = 1$, $\alpha = \frac{1}{2}$, $a = \frac{1}{2}$, vnde erit

$$\frac{dQ}{Q} = \frac{dx(\frac{1}{2} - bx - xx)}{x(\frac{1}{2} - xx)} = \frac{dx(1 - 2bx - 2xx)}{x(1 - 2xx)}$$

siue $\frac{dQ}{Q} = \frac{dx}{x} - \frac{2b\,dx}{1 - 2xx}$

cuius integrale est

$$lQ = lx - \frac{b}{\sqrt{2}}\, l \frac{1 + x\sqrt{2}}{1 - x\sqrt{2}}, \text{ ergo}$$

$$Q = \frac{x(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}}{(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

quae formula euanescit casu $x = \frac{1}{\sqrt{2}}$. Hinc igitur erit

$$dv = \frac{2x(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}dx}{(1 - 2xx)(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

Sit $\frac{b}{\sqrt{2}} = \lambda$ eritque

$$A = 2\int \frac{x(1 - x\sqrt{2})^{\lambda}dx}{(1 - 2xx)(1 + x\sqrt{2})^{\lambda}} = 2\int \frac{x(1 - x\sqrt{2})^{\lambda - 1}dx}{(1 + x\sqrt{2})^{\lambda + 1}}$$

et

$$B = 2\int \frac{xx(1 - x\sqrt{2})^{\lambda - 1}dx}{(1 + x\sqrt{2})^{\lambda + 1}},$$

vbi post integrationem statuitur $x = \frac{1}{\sqrt{2}}$; tum autem fit $s = \frac{A}{B}$, hincque valor fractionis propositae $= b + \frac{B}{A}$.

§. 45.

§. 45. Nisi igitur fuerit $\lambda = \frac{b}{\sqrt{2}}$ numerus rationalis, hos valores commode assignare non licet. Sit igitur $b = \sqrt{2}$, siue $\lambda = 1$, eritque

$$A = 2\int \frac{x\,dx}{(1 + x\sqrt{2})^2} \text{ et } B = 2\int \frac{x\,x\,dx}{(1 + x\sqrt{2})^2}.$$

Hinc integrando colligitur

$$A = l(1 + x\sqrt{2}) - \frac{x\sqrt{2}}{1 + x\sqrt{2}}$$

ideoque posito $x\sqrt{2} = 1$ fiet $A = l\,2 - \frac{1}{2}$; tum vero reperitur

$$B = \frac{3}{2\sqrt{2}} - \sqrt{2}.\,l\,2,$$

quare ob $b = \sqrt{2}$ erit $b + \frac{B}{A} = \frac{1}{\sqrt{2}(2\,l\,2 - 1)}$, vnde sequitur haec summatio:

$$\frac{1}{\sqrt{2}(2\,l\,2 - 1)} = \sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{3}{\sqrt{2} + \cfrac{6}{\sqrt{2} + \text{etc.}}}}$$

Scholion.

§. 46. Fractiones autem continuae, ad quas plerumque calculo numerico deducimur, huiusmodi formam habere solent:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc.}}}}}$$

vbi omnes numeratores sunt vnitates, denominatores vero a, b, c, d, etc. numeri integri. Verum ope nostrae methodi difficulter talium formarum valores eruere licet, etiamsi numeri

meri a, b, c, d, e progressionem arithmeticam constituant, id quod sequenti exemplo ostendamus.

Exemplum.

§. 47. *Proposita sit ista fractio continua:*

$$\beta+b+\cfrac{1}{2\beta+b+\cfrac{1}{3\beta+b+\cfrac{1}{4\beta+b+\cfrac{1}{5\beta+b+\text{etc.}}}}}$$

vbi $\alpha=0$, $\gamma=0$, $a=1$, $c=1$.
Hinc fit

$$\frac{dQ}{Q}=-\frac{dx(1+bx-xx)}{\beta xx}, \text{ vnde}$$

$$lQ=\frac{1}{\beta x}+\frac{b}{\beta}lx+\frac{x}{\beta} \text{ et}$$

$$Q=e^{\frac{1+xx}{\beta x}}.x^{\frac{b}{\beta}},$$

quae autem expressio nullo casu euanescere potest, etiamsi per x^n multiplicetur, siquidem β fuerit numerus positiuus. Verum si pro β sumamus numeros negatiuos puta, $\beta=-m$, tum valor $Q=x^{\frac{-b}{m}}\times e^{\frac{-(1+xx)}{mx}}$, manifesto euanescit, tam si $x=0$, quam si $x=\infty$. Hinc autem erit

$$dv=\frac{x^{\frac{-b}{m}}.e^{\frac{-1-xx}{mx}}dx}{mxx},$$

quamobrem habebimus

$$A=\frac{1}{m}\int\frac{dx}{x^{2+\frac{b}{m}}.e^{\frac{1+xx}{mx}}} \text{ et}$$

B=

$$B = \frac{1}{m}\int \frac{dx}{x^{1+\frac{b}{m}} \cdot e^{\frac{1+xx}{mx}}}.$$

His valoribus inuentis formula $\frac{A}{B}$ exprimet summam huius fractionis continuae:

$$-m+b+1$$
$$\overline{-2m+b+1}$$
$$\overline{-3m+b+1}$$
$$\overline{-4m+b+1}$$
$$\overline{-5m+b+\text{ etc.}}$$

quamobrem formula illa negatiue sumta $-\frac{A}{B}$ exprimet valorem huius fractionis continuae:

$$m+b+1$$
$$\overline{2m-b+1}$$
$$\overline{3m-b+1}$$
$$\overline{4m-b+\text{ etc.}}$$

quem igitur assignare liceret, si modo formulae integrales A et B expediri et a termino $x=0$ ad $x=\infty$ extendi possent. Verum istae formulae ita sunt comparatae, vt earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impedit, quo minus fractio $\frac{A}{B}$ valores satis cognitos inuoluere queat, etiamsi eos nullo adhuc modo assignare valeamus.

§. 49. Talium autem fractionum continuarum mihi quidem binae sequentes innotuere, quarum valores commode exhibere licet:

$n+1$

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}} = \frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1} \text{ et}$$

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}} = \cot.\tfrac{1}{n}$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m-b=n$, $2m-b=3n$, ideoque $m=2n$ et $b=n$, vnde fit

$$A = \tfrac{1}{2n}\int \frac{dx}{x^{\frac{5}{2}}\, e^{\frac{1+xx}{2nx}}} \text{ et}$$

$$B = \tfrac{1}{2n}\int \frac{dx}{x^{\frac{3}{2}}\, e^{\frac{1+xx}{2n}}},$$

vnde iam discimus si hae duae formulae integrentur a termino $x=0$ vsque ad terminum $x=\infty$, tum fore

$$\frac{A}{B} = \frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet, hanc conuenientiam demonstrandi.

SVM-

SVMMATIO
FRACTIONIS CONTINVAE,
CVIVS INDICES PROGRESSIONEM ARITHMETICAM CONSTITVVNT,
DVM NVMERATORES OMNES SVNT VNITATES;
VBI SIMVL RESOLVTIO AEQVATIONIS RICCATIANAE PER HVIVSMODI FRACTIONES DOCETVR.

§. 1.

Cum in praecedente dissertatione methodum exposuissem fractiones continuas ad duas formulas integrales reducendi, ea quidem infinitis casibus feliciter successit: at vero casus, qui simplicissimus videtur, vbi omnes numeratores inter se ponuntur aequales, ad eiusmodi formulas integrales perduxit, quas nullo adhuc modo euoluere et inter se comparare licuit, cum tamen ex hoc genere binae fractiones continuae habeantur, quarum valores satis commode exhiberi possunt:

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\text{etc.}}}} = \frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1} \text{ et}$$

$$\cfrac{n-1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\text{etc.}}}} = \cot.\tfrac{1}{n}$$

quarum quidem altera ex altera facile deducitur, si loco n scribatur $n\sqrt{-1}$.

§. 2. Quando autem indices aliam quamcunque progressionem arithmeticam sequuntur, summationem talium fractionum continuarum iam olim in Tomo XI veterum Commentar. nostrae Academiae singulari prorsus modo ad aequationem Riccatianam reduxi. Methodus autem, qua hoc praestiti, ibi nimis succincte est exposita; quare, cum ea plurimum in recessu habere videatur, eam hic operae pretium erit vberius explicare; praecipue cum non solum nemo vim illius methodi animaduertisse videatur, sed etiam ipse eius penitus essem oblitus.

§. 3. Quo isthanc inuestigationem clarius ob oculos ponam, exordiar a fractione generali, cuius quidem omnes numeratores sint vnitates, indices autem in genere litteris a, b, c, d, e, f, etc. designentur, ita vt ipsa fractio continua hanc habeat formam:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

cuius valor littera S indicetur, ad quem proxime saltem cognoscendum, ex indicibus a, b, c, d, e, etc. formetur more solito series sequentium fractionum:

$a\ b$

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{A}{\mathfrak{A}} & \frac{B}{\mathfrak{B}} & \frac{C}{\mathfrak{C}} & \frac{D}{\mathfrak{D}} & \frac{E}{\mathfrak{E}} \text{ etc.} \end{array}$$

quarum tam numeratores quam denominatores ſequenti modo ex binis praecedentibus determinantur:

$$A = a, \; B = Ab + 1, \; C = Bc + A, \; D = Cd + B, \text{ etc.}$$
$$\mathfrak{A} = 1, \; \mathfrak{B} = \mathfrak{A}b, \quad \mathfrak{C} = \mathfrak{B}c + \mathfrak{A}, \; \mathfrak{D} = \mathfrak{C}d + \mathfrak{B}, \text{ etc.}$$

§. 4. Nunc autem loco litterarum A, B, C, D, et $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, alias in calculum introducamus, quae ſint

$$A' = \frac{A}{a}, \; B' = \frac{B}{ab}, \; C' = \frac{C}{abc}, \; D' = \frac{D}{abcd}, \text{ etc.}$$
$$\mathfrak{A}' = \frac{\mathfrak{A}}{a}, \; \mathfrak{B}' = \frac{\mathfrak{B}}{ab}, \; \mathfrak{C}' = \frac{\mathfrak{C}}{abc}, \; \mathfrak{D}' = \frac{\mathfrak{D}}{abcd}, \text{ etc.}$$

atque hae nouae litterae ſequenti modo per indices et binos antecedentes terminos determinabuntur:

$$A' = 1, \; B' = A' + \frac{1}{ab}, \; C' = B' + \frac{A'}{bc}, \; D' = C' + \frac{B'}{cd}, \text{ etc.}$$
$$\mathfrak{A}' = \frac{1}{a}, \; \mathfrak{B}' = \mathfrak{A}', \quad \mathfrak{C}' = \mathfrak{B}' + \frac{\mathfrak{A}'}{bc}, \; \mathfrak{D}' = \mathfrak{C}' + \frac{\mathfrak{B}'}{cd}, \text{ etc.}$$

His igitur valoribus pro quouis caſu euolutis, iſtae fractiones

$$\frac{A}{\mathfrak{A}}, \frac{B}{\mathfrak{B}}, \frac{C}{\mathfrak{C}}, \frac{D}{\mathfrak{D}}, \frac{E}{\mathfrak{E}} \text{ etc.}$$

continuo propius ad valorem S fractionis continuae propoſitae accedent, et in infinitum continuatae ei prorſus aequabuntur.

§. 5. Quo formae harum litterarum, melius perſpiciantur, eos ſimpliciter per indices euoluamus, ac primo quidem pro numeratoribus reperiemus ſequentes formulas:

$$A' = 1$$
$$B' = 1 + \frac{1}{ab}$$

 C' =

$C' = 1 + \frac{1}{ab} + \frac{1}{bc}$

$D' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{abcd}$

$E' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde}$

$F' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{ef} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde} + \frac{1}{abef} + \frac{1}{bcef} + \frac{1}{cdef} + \frac{1}{abcdef}$

etc. etc.

Pro denominatoribus vero prodibunt ſequentes formulae:

$\mathfrak{A}' = \frac{1}{a}$

$\mathfrak{B}' = \frac{1}{a}$

$\mathfrak{C}' = \frac{1}{a} + \frac{1}{abc}$

$\mathfrak{D}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd}$

$\mathfrak{E}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{abcde}$

$\mathfrak{F}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{aef} + \frac{1}{abcae} + \frac{1}{abcef} + \frac{1}{acdef}$

etc. etc.

in quibus poſterioribus formulis ſinguli termini factorem habent $\frac{1}{a}$, quo omiſſo reliqui factores eodem modo per indices b, c, d, e, f, etc. definientur, quo litterae latinae antecedentes per omnes indices a, b, c, d, e ſunt determinati.

§. 6. Accommodemus nunc iſtas euolutiones ad caſum qui hic nobis eſt propoſitus, vbi indices a, b, c, d, e, ſecundum progreſſionem arithmeticam procedere aſſumimus. Statuamus igitur differentiam, qua hi indices continuo creſcunt $= \Delta$, eruntque indices poſt primum ſequentes

$$b = a + \Delta,\ c = a + 2\Delta,\ d = a + 3\Delta,\ e = a + 4\Delta, \text{ etc.}$$

Hos quidem valores in denominatoribus noſtrarum formularum non ſubſtituemus, ſed iis praecique vtemur ad formulas contrahendas.

§. 7.

§. 7. Hac igitur progreſſione ſtabilita euoluamus primo numeratores noſtrarum fractionum ſequenti modo:

$$A' = 1$$

$$B' = 1 + \frac{1}{ab}$$

$$C' = 1 + \frac{1}{ab} + \frac{1}{bc} = 1 + \frac{2}{ac}$$

$$D' = 1 + \frac{2}{ac} + \frac{1}{cd} + \frac{1}{abcd} = 1 + \frac{3}{ad} + \frac{1}{abcd}$$

$$E' = 1 + \frac{3}{ad} + \frac{1}{de} + \frac{1}{abcd} + \frac{2}{acde} = 1 + \frac{4}{ae} + \frac{3}{abde}$$

$$F' = 1 + \frac{5}{af} + \frac{6}{abaf} + \frac{1}{abcdef}$$

$$G' = 1 + \frac{6}{ag} + \frac{10}{abfg} + \frac{4}{abcefg}$$

$$H' = 1 + \frac{7}{ah} + \frac{15}{abgh} + \frac{10}{abcfgh} + \frac{1}{abcdefgh}$$

$$I' = 1 + \frac{8}{ai} + \frac{21}{abhi} + \frac{20}{abcghi} + \frac{5}{abcdfghi}$$

$$K' = 1 + \frac{9}{ak} + \frac{28}{abik} + \frac{35}{abchik} + \frac{15}{abcdghik} + \frac{1}{abcdefghik}.$$

§. 8. Quemadmodum in his formis omnes termini primi ſunt vnitates, ita numeratores ſecundorum ſecundum numeros naturales, tertiorum ſecundum trigonales, quartorum ſecundum pyramidales primos, tum ſecundos, tertios etc. progrediuntur. In denominatoribus ordo pariter ſatis eſt manifeſtus. Hinc igitur in genere eam formulam exhibere poterimus, quae indefinite reſpondeat indici i. Ita ſi iſta expreſſio deſignetur littera Z', tum vero in ordine litterarum $a, b, c, d, e,$ etc. fuerit $z = a + i\Delta$, antecedentes vero

$$y = a + (i-1)\Delta,\quad x = a + (i-2)\Delta,\quad v = a + (i-3)\Delta, \text{ etc.}$$

habebimus

$$Z' = 1 + \frac{i}{az} + \frac{(i-1)(i-2)}{1.2.ab.yz} + \frac{(i-2)(i-3)(i-4)}{1.2.3.abcxyz} + \frac{(i-3)(i-4)(i-5)(i-6)}{1.2.3.4.abcdvxyz} + \text{etc.}$$

§. 9. Quod iam ad litteras germanicas $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}'$, etc. attinet, earum quaelibet ex antecedente latina formatur, dum litterae a, b, c, d, e vno gradu promouentur, tum vero singuli termini per $\frac{1}{a}$ multiplicentur, ita erit vt sequitur:

$$\mathfrak{A}' = 1$$

$$\mathfrak{B}' = \frac{1}{a}$$

$$\mathfrak{C}' = \frac{1}{a} + \frac{1}{abc}$$

$$\mathfrak{D}' = \frac{1}{a} + \frac{2}{abd}$$

$$\mathfrak{E}' = \frac{1}{a} + \frac{3}{abe} + \frac{1}{abcde}$$

$$\mathfrak{F}' = \frac{1}{a} + \frac{4}{abf} + \frac{3}{abcef}$$

$$\mathfrak{G}' = \frac{1}{a} + \frac{5}{abg} + \frac{6}{abcfg} + \frac{1}{abcdefg}$$

$$\mathfrak{H}' = \frac{1}{a} + \frac{6}{abh} + \frac{10}{abchi} + \frac{4}{abcdfgh}$$

$$\mathfrak{I}' = \frac{1}{a} + \frac{7}{abi} + \frac{15}{abchi} + \frac{10}{abcdghi} + \frac{1}{abcdefghi}$$

etc. etc.

vnde in genere, si indici i respondeat $\mathfrak{Z}'$, erit

$$\mathfrak{Z}' = \frac{1}{a} + \frac{(i-1)}{abz} + \frac{(i-2)(i-3)}{1.2.abcyz} + \frac{(i-3)(i-4)(i-5)}{1.2.3.abcdxyz} + \frac{(i-4)(i-5)(i-6)(i-7)}{1.2.3.4.abcdevxyz} + \text{etc.}$$

§. 10. Augeamus nunc indicem z in infinitum, vt fractio $\frac{Z'}{\mathfrak{Z}'}$ exprimat ipsum valorem fractionis continuae, quem vocauimus S, ita vt futurum sit $S = \frac{Z'}{\mathfrak{Z}'}$, atque reductio formularum inuentarum sequenti modo peragetur. Primo scilicet pro Z' erit $\frac{i}{z} = \frac{i}{a + i\Delta} = \frac{1}{\Delta}$, ob $i = \infty$; pro termino tertio erit

$$\frac{(i-1)(i-2)}{yz} = \frac{(i-1)(i-2)}{(a + (i-1)\Delta)(a + i\Delta)} = \frac{1}{\Delta^2};$$

porro

porro autem ſimili modo

$$\frac{(i-2)(i-3)(i-4)}{x\,y\,z} = \frac{(i-2)(i-3)(i-4)}{(a+(i-2)\Delta)(a+(i-1)\Delta)(a+i\Delta)} = \frac{1}{\Delta^3} \text{ et}$$

$$\frac{(i-3)(i-4)(i-5)(i-6)}{v\,x\,y\,z} = \frac{1}{\Delta^4}, \text{ etc.}$$

Pari modo pro formula $\mathfrak{Z}'$ erit etiam

$$\frac{i-1}{z} = \frac{1}{\Delta}, \quad \frac{(i-2)(i-3)}{y\,z} = \frac{1}{\Delta^2}$$

et ita porro; quamobrem pro caſu $i = \infty$ ambae noſtrae formulae ita commode contrahuntur, vt fiat

$$Z' = 1 + \frac{1}{a\Delta} + \frac{1}{1.2\,ab\,\Delta^2} + \frac{1}{1.2.3\,abc\,\Delta^3} + \frac{1}{1.2.3.4\,abcd\,\Delta^4} + \text{ etc.}$$

ſimiliqne modo

$$\mathfrak{Z}' = \frac{1}{a} + \frac{1}{ab\,\Delta} + \frac{1}{1.2\,abc\,\Delta^2} + \frac{1}{1.2.3\,abcd\,\Delta^3} + \frac{1}{1.2.3.4\,abcde\,\Delta^4} + \text{ etc.}$$

§. 11. Cum igitur ſit valor quaeſitus $S = \frac{Z'}{\mathfrak{Z}'}$, videamus quomodo ambas ſeries infinitas inuentas ad expreſſiones finitas reuocare queamus. Hunc in finem ambas ſeries generaliores reddamus, dum loco numeratorum, qui omnes ſunt 1, progreſſionem quandam geometricam ſubſtituimus. Statuamus igitur

$$p = 1 + \frac{x^\Delta}{a\Delta} + \frac{x^{2\Delta}}{1.2\,ab\Delta^2} + \frac{x^{3\Delta}}{1.2.3\,abc\Delta^3} \text{ etc. et}$$

$$q = \frac{1}{a} + \frac{x^\Delta}{ab\Delta} + \frac{x^{2\Delta}}{1.2\,abc\Delta^2} + \frac{x^{3\Delta}}{1.2.3\,abcd\Delta^3} \text{ etc.}$$

Quod ſi autem hinc valores p et q eruerimus in genere, tum vero ponamus $x = 1$, vtique proueniet $S = \frac{p}{q}$. Hic autem manifeſtum eſt, ambas has ſeries egregiam inter ſe tenere affinitatem, ac per differentiationem vnam in alteram conuerti poſſe, quam inueſtigationem ſequenti modo inſtituamus.

§. 12.

§. 12. Primo igitur feries prior fimpliciter differentiata dat

$$\frac{x\,dp}{dx} = \frac{x^{\Delta}}{a} + \frac{x^{2\Delta}}{ab\Delta} + \frac{x^{3\Delta}}{1.\,2\,abc\Delta^2} + \frac{x^{4\Delta}}{1.\,2.\,3\,abcd\Delta^3} \text{ etc.}$$

quae feries cum altera q comparata manifefto praebet

$$\frac{x\,dp}{dx} = x^{\Delta}\,q,$$

vnde patet, fi modo fumma alterius harum duarum feriérum effet cognita, alteram quoque affignari poffe, quandoquidem ex cognito valore p prodit $q = \frac{dp}{x^{\Delta-1}dx}$; contra vero ex cognito valore q fit $dp = x^{\Delta-1}\,q\,dx$, ideoque $p = \int x^{\Delta-1}\,q\,dx$, quod integrale ita fumi debet, vt pofito $x = 0$ fiat $p = 1$.

§. 13. Ante autem quam alteram feriem differentiemus, eam multiplicemus per x, atque ob

$$a + \Delta = b,\ a + 2\Delta = c,\ a + 3\Delta = d, \text{ etc. erit}$$

$$x^a q = \frac{x^a}{a} + \frac{x^b}{ab\Delta} + \frac{x^c}{1.\,2\,abc\Delta^2} + \frac{x^d}{1.\,2.\,3\,abcd\Delta^3} + \text{ etc.}$$

Nunc ifta aequatio differentiata, iterumque per x multiplicata dabit

$$x\,d.\,\frac{x^a q}{dx} = x^a + \frac{x^b}{a\Delta} + \frac{x^c}{1.\,2\,ab\Delta^2} + \frac{x^d}{1.\,2.\,3\,abc\Delta^3} + \text{ etc.}$$

at vero prior feries p itidem qer x multiplicata praebet

$$x^a p = x^a + \frac{x^b}{a\Delta} + \frac{x^c}{1.\,2\,ab\Delta^2} + \frac{x^d}{1.\,2.\,3\,abc\Delta^3} + \text{ etc.}$$

quae

quae series cum sint perfecte aequales, erit $\frac{x}{dx} d. x^a q = x^a p$, ideoque $d. x^a q = p x^{a-1} dx$, sicque duas nacti sumus aequationes differentiales inter p et q, ex quibus valorem vtriusque eruere licebit.

§. 14. Cum ex priore aequatione sit

$$q = \frac{dp}{x^{\Delta-1} dx}, \text{ erit } x^a q = \frac{x^{a-\Delta+1} dp}{dx},$$

vnde sumto elemento dx constante fiet

$$d. x^a q = \frac{x^{a-\Delta+1} ddp + (a-\Delta+1) x^{a-\Delta} dp}{dx}$$

sicque elisa quantitate q pro altera p nanciscimur hanc aequationem differentialem secundi gradus:

$$p x^{a-1} dx = \frac{x^{a-\Delta+1} ddp + (a-\Delta+1) x^{a-\Delta} dp}{dx}$$

quam si resoluere licuerit, totum negotium erit confectum. Vbi probe notandum, cum sit

$$p = 1 + \frac{x^\Delta}{a\Delta} + \frac{x^{2\Delta}}{1.2 ab \Delta^2} + \text{etc.}$$

integrationem ita institui debere, vt posito $x = 0$ fiat $p = 1$; tum vero quia est

$$\frac{dp}{dx} = \frac{x^{\Delta-1}}{a} + \frac{x^{2\Delta-1}}{ab\Delta} + \text{etc.}$$

altera conditio integrationis postulat vt posito $x = 0$ etiam fiat $\frac{dp}{dx} = 0$, siquidem fuerit $\Delta > 1$, si enim esset $= 1$, casu $x = 0$ fieri debet $\frac{dp}{dx} = \frac{1}{a}$. At si $\Delta < 1$, fieri debebit $\frac{dp}{dx} = \infty$.

§. 15. Cum fractio $\frac{p}{q}$ posito $x = 1$ praebeat valorem nostrae fractionis continuae S, ponamus in genere esse $\frac{p}{q} = z$, ita vt posito $x = 1$ fiat $z = S$, vnde patet, casu $x = 0$ fieri debere $z = a$. Cum igitur sit $p = qz$; erit $dp = q\,dz + z\,dq$; erat autem ex prima aequatione

$$q = \frac{dp}{x^{\Delta-1}dx},$$

quamobrem habebimus

$$q = \frac{q\,dz + z\,dq}{x^{\Delta-1}dx},$$

siue $x^{\Delta-1}q\,dx = q\,dz + z\,dq$, vnde fit

$$dq = \frac{x^{\Delta-1}q\,dx - q\,dz}{z};$$

erat autem $dp = x^{\Delta-1}q\,dx$.

§. 16. Haec igitur sequuntur ex priori aequatione differentiali inuenta $\frac{dp}{x^{\Delta-1}dx} = q$. Altera vero, quae est

$$d.\,x^a q = p\,x^{a-1}dx, \text{ quia}$$
$$d.\,x^a q = a\,x^{a-1}q\,dx + x^a dq$$

loco dq posito valore modo inuento prodit

$$d.\,x^a q = a x^{a-1}q\,dx + \frac{x^{a+\Delta-1}q\,dx - x^a q\,dz}{z},$$

habebimus

$$p x^{a-1}dx = a x^{a-1}q\,dx + \frac{x^{a+\Delta-1}q\,dx - x^a q\,dz}{z},$$

quae

quae aequatio, ob $p = qz$, per q diuisa suppeditat istam aequationem differentialem primi gradus:

$$x^{a-1} z dx = a x^{a-1} dx + \frac{x^{a+\Delta-1} dx - x^a dz}{z},$$

quae per z multiplicata et per x^{a-1} diuisa praebet

$$zz dx = az dx + x^\Delta dx - x dz,$$

cuius ergo resolutio si ita instituatur, vt posito $x = 0$ fiat $z = a$, tum vero fiat $x = 1$, valor ipsius z dabit ipsum valorem fractionis continuae quem quaerimus.

§. 17. Totum igitur negotium perductum est ad resolutionem aequationis differentialis primi gradus

$$zz dx = az dx + x^\Delta dx - x dz,$$

quae manifesto in celeberrima illa aequatione Riccatiana continetur. Vt enim ad tres tantum terminos reducatur, ponatur $z = x^a y$, ita vt sit $y = \frac{z}{x^a}$, vnde fit

$$dz = a x^{a-1} y dx + x^a dy,$$

quibus valoribus substitutis nostra aequatio hanc induet formam:

$$x^{a+1} dy + x^{2a} yy dx = x^\Delta dx,$$

quam ergo ita integrari oportet, vt posito $x = 0$, siue infinite paruo, fiat $y = \frac{a}{x^a}$, hoc est $y = \infty$, quo facto, si post integrationem statuatur $x = 1$, valor ipsius y dabit summam fractionis continuae propositae.

§. 18,

§. 18. Quo hanc expressionem simpliciorem reddamus, diuidamus per x^{a+1}, vt habeamus

$$dy + x^{a-1}yy\,dx = x^{\Delta-a-1}\,dx;$$

nunc vero statuamus $x^a = t$, ita vt casu $x = 0$ fiat quoque $t = 0$, et casu $x = 1$ etiam $t = 1$, vnde si t euanescat, fieri debet $y = \frac{a}{t}$, siue $y = \infty$. Hoc autem valore introducto, ob $x = t^{\frac{1}{a}}$ et

$$dx = \frac{t^{\frac{1-a}{a}}\,dt}{a},$$

aequatio nostra fiet

$$a\,dy + yy\,dt = t^{\frac{\Delta-2a}{a}}\,dt,$$

quae est forma maxime vsitata aequationis Riccatianae.

§. 19. Quando ergo proposita fuerit talis fractio continua:

$$a + \cfrac{1}{a + \Delta + \cfrac{1}{a + 2\Delta + \cfrac{1}{a + 3\Delta + \cfrac{1}{a + 4\Delta + \text{etc.}}}}}$$

pro eius valore inuestigando resolui debet ista aequatio Riccatiana:

$$a\,dy + yy\,dt = t^{\frac{\Delta-2a}{a}}\,dt;$$

vbi integrationem ita institui oportet, vt sumta t infinite parua fiat $y = \frac{a}{t}$, quo facto statuatur $t = 1$, et valor pro y resultans erit valor huius fractionis continuae.

§. 20

§. 20. Euoluamus casum simplicissimum, quo ad dextram partem exponens ipsius t fit nihilo aequalis, quod ergo euenit, si $\Delta = 2a$, ideoqne ipsa fractio continua

$$a+1 \cfrac{}{3a+1 \cfrac{}{5a+1 \cfrac{}{7a+ \text{etc.}}}}$$

pro cuius summa habebimus hanc aequationem differentialem:

$$a\,dy + yy\,dt = dt, \text{ vnde } dt = \frac{a\,dy}{1-yy},$$

et integrando

$$t = \frac{a}{2} l \frac{1+y}{1-y} + C,$$

quae constans C ita est capienda, vt posito $t = 0$ fiat $y = \frac{a}{t}$, ideoque $t = \frac{a}{y}$; vnde patet hoc casu fieri $y = \infty$, ex quo statim intelligitur, aequationem integralem ita instrui debere:

$$t = \frac{a}{2} l \frac{y+1}{y-1} + C.$$

siue integrationem ita institui, vt facto $t = 0$ fiat y infinitum. Nunc vero si y infinitum, erit $l\frac{y+1}{y-1} = \frac{2}{y}$, quare cum fieri debeat $t = \frac{a}{y}$ fit $C = 0$, ita vt iusta aequatio integralis fit $t = \frac{a}{2} l \frac{y+1}{y-1}$, vnde, denotante e numerum cuius logarithmus hyperbolicus $= 1$, fiet $e^{\frac{2t}{a}} = \frac{y+1}{y-1}$, hincque porro $y = \frac{e^{\frac{2t}{a}}+1}{e^{\frac{2t}{a}}-1}$, vnde posito $t = 1$ summa nostrae fractionis continuae erit $\frac{e^{\frac{2}{a}}+1}{e^{\frac{2}{a}}-1}$, qui est idem valor, quem iam dudum inueneram.

§. 21. Contemplemur nunc etiam reliquos casus integrabilitatis aequationis Riccatianae, quibus exponens ipsius t ad partem dextram est vel -4, vel $-\frac{4}{3}$, vel $-\frac{8}{3}$, vel $-\frac{8}{5}$, vel $-\frac{12}{5}$, vel $-\frac{12}{7}$ etc. Sit igitur primo $\frac{\Delta-2a}{a}=-4$, vel $\Delta=-2a$, vnde nascitur haec fractio continua:

$$a+\cfrac{1}{-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\cfrac{1}{-7a+\text{etc.}}}}}$$

quae manifesto a praecedente pendet. Si enim ponamus

$$-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\text{etc.}}}=s,$$

mutatis signis erit

$$-s=a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

cuius valor cum iam sit inuentus, iste casus nihil noui nobis offert.

§. 22. Si sumatur $\frac{\Delta-2a}{a}=-\frac{4}{3}$, siue $\Delta=\frac{2}{3}a$, inde sequens nascitur fractio continua:

$$a+\cfrac{1}{\frac{5}{3}a+\cfrac{1}{\frac{7}{3}a+\cfrac{1}{\frac{9}{3}a+\text{etc.}}}}$$

siue

sive statuendo $a = 3\alpha$, erit fractio

$$3\alpha + \cfrac{1}{5\alpha + \cfrac{1}{7\alpha + \cfrac{1}{9\alpha + \text{etc.}}}}$$

quae est ipsa ipsa forma superior primo membro truncata. Idem vsu venit, si sumatur $\frac{\Delta - 2a}{a} = -\frac{8}{3}$, siue $\Delta = -\frac{2}{3}a$, vnde oritur haec fractio continua, ponendo scil. $a = 3\alpha$

$$3\alpha + \cfrac{1}{\alpha + \cfrac{1}{-\alpha + \cfrac{1}{-3\alpha + \cfrac{1}{-5\alpha + \text{etc.}}}}}$$

sicque semper ad principalem formam reducimur.

§. 23. Casus autem integrabilitatis in genere hoc continentur exponente: $-\frac{4i}{2i \pm 1}$. Posito igitur

$$\frac{\Delta - 2a}{a} = -\frac{4i}{2i \pm 1}, \text{ fiet } \Delta = \frac{\pm 2a}{2i \pm 1},$$

vnde posito $\frac{a}{2i \pm 1} = \alpha$, erit $\Delta = \pm 2\alpha$, ergo ob $a = (2i \pm 1)\alpha$ fractio continua erit

$$(2i \pm 1)\alpha + \cfrac{1}{a \pm 2\alpha + \cfrac{1}{a \pm 4\alpha + \cfrac{1}{a \pm 6\alpha + \text{etc.}}}}$$

vbi manifesto iterum omnes numeri impares occurrunt, ita vt fractio continua inde nata semper formari possit ex nostra principali

$$a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

si vel aliquot membris truncetur, vel retro ad aliquot membra superne continuetur.

§. 24. Cum igitur omnes casus integrabilitatis aequationis Riccatianae ad eandem fractionem continuam perducant; praeterea vero nulli adhuc alii casus euolui potuerint, hinc manifesto sequitur, si indices fractionis continuae a, b, c, d, e, f etc. quamcunque aliam progressionem arithmeticam constituant, tum summam nullo plane modo assignari posse; quandoquidem ea pendet a casu irresolubili aequationis Riccatianae. Ita si proponatur haec fractio continua:

$$a+\cfrac{1}{2a+\cfrac{1}{3a+\cfrac{1}{4a+\text{etc.}}}}$$

vbi est $\Delta = a$, summatio pendebit ab ista aequatione differentiali: $a\,d\,y + y\,y\,d\,t = \frac{d\,t}{t}$, cuius resolutio cum per nullas quantitates transcendentes etiamnunc vsu receptas expediri possit, valorem huius fractionis continuae frustra inter quantitates a circulo vel a logarithmis pendentes, vel adeo inter omnes quadraturas curuarum algebraicarum quaerimus; vnde mirum non est, quod methodus in superiori dissertatione vsitata pro talibus casibus omni successu caruerit.

§. 25. Quoniam tamen in dissertatione superiore summam talium fractionum continuarum per binas formulas inte-

integralas expressam dedimus, illa ipsa expressio etiam ad aequationem Riccatianam pro iisdem casibus accommodari poterit, id quod vtique maximam attentionem meretur, cum nullo adhuc modo ista aequatio praeter casus integrabiles tractari potuerit. Quamobrem maxime operae erit pretium solutiones inter se comparare, quandoquidem hinc haud contemnendum subsidium, aequationem Riccatianam feliciori successu tractandi, expectari poterit.

§. 26. In superiori autem dissertatione ostendi, si haec proposita fuerit fractio continua:

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

eius valorem exprimi per hanc fractionem: $-\frac{A}{B}$, existente

$$A=\int\frac{dx}{x^{2+\frac{b}{m}}.e^{\frac{1+xx}{mx}}} \text{ et}$$

$$B=\int\frac{dx}{x^{1+\frac{b}{m}}.e^{\frac{1+xx}{mx}}},$$

siquidem haec duo integralia a termino $x=0$ vsque ad terminum $x=\infty$ extendantur.

§. 27. Comparemus nunc hanc fractionem continuam cum generali, quam hic tractauimus:

$$a+1\cfrac{}{a+\Delta+1\cfrac{}{a+2\Delta+1\cfrac{}{a+3\Delta+\text{ etc.}}}}$$

cuius valor continetur in hac aequatione:

$$a\,dy+yy\,dt=t^{\frac{\Delta-2a}{a}}\,dt,$$

integratione scilicet ita instituta, vt casu $t=0$ fiat $y=\infty$; tum vero statuatur $t=1$, vnde valor ipsius y summam istius fractionis continuae exprimet.

§. 28. Comparatione igitur instituta fiet $m=\Delta$ et $b=\Delta-a$, quibus valoribus introductis formulae illae integrales erunt

$$A=\int\frac{dx}{x^{3-\frac{a}{\Delta}}\,e^{\frac{1+xx}{\Delta x}}}\text{ et}$$

$$B=\int\frac{dx}{x^{2-\frac{a}{\Delta}}\,e^{\frac{1+xx}{\Delta x}}},$$

integralibus iterum sumtis ab $x=0$ ad $x=\infty$; quamobrem valor ipsius y, qui ex aequatione

$$a\,dy+yy\,dt=t^{\frac{\Delta-2a}{a}}\,dt$$

pro casu $t=1$ resultat, aequalis erit isti fractioni: $-\frac{A}{B}$, siue erit

$$y=\frac{\int dx:x^{3-\frac{a}{\Delta}}\,e^{\frac{1+xx}{\Delta x}}}{\int dx:x^{2-\frac{a}{\Delta}}\,e^{\frac{1+xx}{\Delta x}}},$$

Quanquam autem haec aequalitas tantum casus speciales, quibus ibi fit $t=1$, hic vero $x=\infty$, spectat, tamen for-

taſſe

talis eiusmodi relatio inter binas variabiles t et v inueniri poterit, vt in genere quantitas y illi formulae aequetur.

§. 29. Quemadmodum consideratio fractionis nostrae continuae nos ad resolutionem aequationis Riccatianae perduxit, ita vicissim datur methodus directa, qua ista aequatio per fractiones continuas resolui potest. Quod quo facilius ostendatur, aequatio Riccatiana, quae vulgo hac forma proponi solet:

$$dy + ayy\,dx = acx^{2m}\,dx,$$

in aliam formam ad praesens institutum magis accommodatam transfundatur, ponendo $y = x^m z$ et $x^{m+1} = t$; tum enim, si loco $\frac{a}{m+1}$ scribatur b, prodit ista aequatio:

$$dz + \frac{mz\,dt}{(m+1)t} + bzz\,dt = bc\,dt.$$

§. 30. Ponamus porro $\frac{m}{m+1} = -n$, vt sit $m = -\frac{n}{1+n}$, et nobis proposita sit ista aequatio:

$$dz - n\frac{z\,dt}{t} + bzz\,dt = bc\,dt,$$

quae, si adhibeamus hanc substitutionem: $z = \frac{n+1}{bt} + \frac{c}{v}$, transmutatur in hanc formam:

$$dv - (n+2)\frac{v\,dt}{t} + bvv\,dt = bc\,dt,$$

quae a priore hoc tantum differt, vt hic numerus n binario maior sit factus; quare si porro faciamus $v = \frac{n+3}{bt} + \frac{c}{u}$, prodibit haec aequatio:

$$du - (n+4)\,u\,dt + buu\,dt = bc\,dt,$$

vnde patet, si prima aequatio resolutionem admittat casu $n = k$, tum etiam eius resolutionem in potestate fore casibus

$$n = k+2,\ n = k+4,\ n = k+6,\ \text{etc.}$$

 et

et in genere casu $n = k + 2i$, denotante i numerum integrum quemcunque.

§. 31. Quod si iam successiue loco v et u et sequentium litterarum valores istos debitos substituamus, pro quantitate z sequens prodibit fractio continua:

$$z = n + \cfrac{1}{bt + \cfrac{c}{\frac{n+1}{bt} + \cfrac{c}{\frac{n+3}{bt} + \text{etc.}}}}$$

quae ergo expressio exhibet valorem quantitatis z, qui ipsi conuenit vi huius aequationis:

$$dz - n\frac{z\,dt}{t} + bzz\,dt = bc\,dt,$$

si scilicet ita integretur, vt posito $t = 0$ fiat $z = \infty$.

§. 32. Liberemus hanc formam a fractionibus partialibus, et obtinebimus hanc formam:

$$\frac{1}{z} = \cfrac{bt}{n+1 + \cfrac{bbctt}{n+3 + \cfrac{bbctt}{n+5 + \cfrac{bbctt}{n+7 + \text{etc.}}}}}$$

vnde statim patet, posito $t = 0$ fore $\frac{1}{z} = 0$ ideoque $z = \infty$. Simili modo resolutionem ab aequatione transformata exordiri possumus, quae erat

$$dv - (n+2)\frac{v\,dt}{t} + bvv\,dt = bc\,dt,$$

quae in primam transformatur ponendo $v = \frac{bct}{-n-1+btz}$; prodibit enim

dz

$$dz - \frac{nzdt}{t} + bzzdt = bcdt,$$

in qua aequatione numerus n binario redditus est minor; vnde patet, si aequatio resolutionem admittat casu $n = k$, tum etiam resolutionem esse successuram casibus

$$n = k - 2, \; n = k - 4, \; n = k - 6$$

et in genere $n = k - 2i$. Quare cum resolutio nulla laboret difficultate casu $n = 0$, sumto $k = 0$ omnes casus resolutionem admittentes continebuntur in hac formula: $n = \pm 2i$ hincque pro forma consueta fiet

$$m = \frac{\mp 2i}{\pm 2i + 1} = -\frac{2i}{2i \pm 1},$$

quae continet casus notissimos integrabilitatis.

§. 33. Ponamus $n + 2 = -\nu$, vt aequatio, a qua hic inchoamus, fit

$$dv = \frac{\nu v dt}{t} + bvvdt = bcdt,$$

quae ergo, posito $v = \frac{bct}{\nu + 1 + btz}$ transmutatur in hanc formam:

$$dz + (\nu + 2)\frac{zdt}{t} + bzzdt = bcdt,$$

in qua nunc numerus ν binario augetur. Quare si vlterius ponamus $z = \frac{bct}{\nu + 3 + bty}$, orietur haec aequatio:

$$dy = + (\nu + 4)\frac{ydt}{t} + byydt = bcdt,$$

sicque vlterius progrediendo peruenietur ad $\nu + 5$, $\nu + 7$ etc.

§. 34. Substituamus ergo istos valores in superiore aequatione, quae est

$$dv + \frac{\nu v dt}{t} + bvvdt = bcdt,$$

ac pro v prodibit sequens fractio continua:

 $v =$

$$v = \cfrac{bct}{\nu + 1 + \cfrac{bbctt}{\nu + 3 + \cfrac{bbctt}{\nu + 5 + \cfrac{bbctt}{\nu + 7 + \text{etc.}}}}}$$

Haec igitur expressio locum habet, si aequatio differentialis ita integretur, vt posito $t = 0$ fiat $v = 0$.

§. 35. Geminas igitur ex aequatione Riccatiana elicuimus fractiones continuas, quas quo facilius inter se comparare queamus, loco ν scribamus iterum n, vt habeamus has duas aequationes differentiales:

$$\text{I. } dz - n\frac{z\,dt}{t} + bzz\,dt = bc\,dt,$$

$$\text{II. } dv + n\frac{v\,dt}{t} + bvv\,dt = bc\,dt,$$

atque ex priore nascetur ista fractio continua:

$$\frac{1}{z} = \cfrac{bt}{n + 1 + \cfrac{bbctt}{n + 3 + \cfrac{bbctt}{n + 5 + \text{etc.}}}}$$

ex altera vera oritur

$$v = \cfrac{bct}{n + 1 + \cfrac{bbctt}{n + 3 + \cfrac{bbctt}{n + 5 + \text{etc.}}}}$$

quae ergo fractiones prorsus inter se conueniunt, cum hinc fiat $v = \frac{c}{z}$, quippe quo modo altera aequatio in alteram actu conuertitur, ita vt hae duae formae pro vnica sint habendae.

§. 36.

§. 36. Ista resolutio aequationis Riccatianae in fractionem continuam, eo magis est attentione digna, quod haec aequatio nullo adhuc modo in seriem infinitam regularem resolui potuerit. Quae enim series olim pro eius resolutione exhibui, ita sunt comparatae vt vna series infinita per aliam diuisa resolutionem aequationis Riccatianae suppeditet; hic autem de vnica serie simplici sermo instituitur. Hinc igitur nascitur quaestio, num forte non etiam aliae aequationes differentiales dentur, quarum resolutionem pariter per fractiones continuas expedire liceat.

DE SVMMA SERIEI EX NVMERIS PRIMIS FORMATAE

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31} \text{ etc.}$$

VBI NVMERI PRIMI FORMAE $4n-1$ HABENT SIGNVM POSITIVVM, FORMAE AVTEM $4n+1$ SIGNVM NEGATIVVM.

§. 1.

Cum iam *Euclides* demonstrasset, multitudinem numerorum primorum reuera esse infinitam, ego iam pridem ostendi etiam summam seriei reciprocae numerorum primorum: scilicet

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+ \text{ etc.}$$

esse infinite magnam, atque adeo referre logarithmum summae seriei harmonicae

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+ \text{ etc.}$$

id quod non parum mirum videbatur, cum vulgo summa seriei harmonicae ad genus quasi infimum infinitorum referri soleat. Cum autem non solum logarithmus numeri infiniti sit etiam infinitus, sed etiam logarithmi horum ipsorum logarithmorum etiamnunc sint infiniti; manifestum est dari insuper infinitos gradus inferiores infinitorum. Ita si A denotet summam seriei reciprocae numerorum primorum, etiam

l A

lA adhuc erit infinite magnus, ſed ad ordinem infinitorum infinities inferiorem pertinere cenſendus eſt; tum vero etiamnunc hae formulae: llA, $lllA$, $llllA$, etc. erunt infinitae, quanquam quaelibet earum infinities ſit minor quam praecedens.

§. 2. Quoniam porro numeri primi praeter binarium quaſi a natura in duas claſſes diſtinguuntur, prouti fuerint vel formae $4n+1$, vel formae $4n-1$; dum priores omnes ſunt ſummae duorum quadratorum, poſteriores vero ab hac proprietate penitus excluduntur: ſeries reciprocae ex vtraque claſſe formatae, ſcilicet:

$$\tfrac{1}{5}+\tfrac{1}{13}+\tfrac{1}{17}+\tfrac{1}{29}+ \text{ etc. et}$$

$$\tfrac{1}{3}+\tfrac{1}{7}+\tfrac{1}{11}+\tfrac{1}{19}+\tfrac{1}{23}+ \text{ etc.}$$

ambae erunt pariter infinitae, id quod etiam de omnibus ſpeciebus numerorum primorum eſt tenendum. Ita ſi ex numeris primis ii tantum excerpantur, qui ſunt formae $100n+1$, cuiusmodi ſunt 101, 401, 601, 701 etc., non ſolum multitudo eorum eſt infinita, ſed etiam ſumma huius ſeriei ex illis formatae, ſcilicet:

$$\tfrac{1}{101}+\tfrac{1}{401}+\tfrac{1}{601}+\tfrac{1}{701}+\tfrac{1}{1201}+\tfrac{1}{1301}+\tfrac{1}{1601}+\tfrac{1}{1801}+\tfrac{1}{1901}+ \text{ etc.}$$

etiam eſt infinita.

§. 3. Conſideremus hic autem imprimis diſcrimen inter numeros primos formae $4n+1$ et $4n-1$, et quia ambae ſeries ex vtroque ordine formatae ſunt infinitae et quaſi eiusdem ordinis; nullum eſt dubium, quin earum differentia habeat valorem determinatum. Hanc ob rem terminis ex forma $4n-1$ formatis tribuamus ſignum $+$, reliquis vero ſignum $-$, vt orietur iſta ſeries:

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}-\frac{1}{17}+\frac{1}{19}+\frac{1}{23}-\frac{1}{29}+\frac{1}{31} \text{ etc},$$

in qua nullus plane ordo in ſignis cernitur. Hoc tamen non obſtante videamus, quomodo eius ſumma vero ſaltem proxime aſſignari queat. Non parum enim probabile videtur, iſtam ſummam, ſi non fuerit rationalis vel irrationalis, ſaltem ad quodpiam genus notabile quantitatum tranſcendentium pertinere debere. Interim tamen, quoniam non ſolum non in ſignis, verum etiam multo minus in ipſis fractionibus nullus plane ordo perſpicitur, primo intuitu nulla patere videtur via, qua ad eius ſummam peruenire liceat.

§. 4. Quando autem contemplamur ſeriem notiſſimam Leibnitzianam pro quadratura circuli datam

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11} \text{ etc.}$$

videmus, in ea omnes numeros impares formae $4n+1$ habere ſignum $+$, reliquos vero formae $4n-1$ ſignum $-$, vnde mutatis ſignis omnes termini noſtrae ſeriei cum ſuis ſignis in hac ſerie Leibnitziana occurrunt. Quod ſi ergo ex hac ſerie omnes numeros compoſitos expungamus, tandem praeter vnitatem ipſa noſtra ſeries propoſita, contrariis ſignis, remanebit. Quamobrem ſummam noſtrae ſeriei obtinebimus, ſi ex ſerie Leibnitziana ſucceſſiue omnes numeros compoſitos excludamus, quandoquidem omnium terminorum, qui per quamuis operationem excluduntur, ſumma facile aſſignare licebit.

§. 5. Incipiamus igitur ab ipſa ſerie Leibnitziana, ſtatuendo

$$A=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+ \text{ etc.}$$

ita

ita vt ſit $A = \frac{\pi}{4}$, ex eaque primo omnes numeros compoſitos per 3 diuiſibiles excludamus. Quem in finem hinc formemus iſtam ſeriem:

$$(A-1)\frac{1}{3} = -\frac{1}{9}+\frac{1}{15}-\frac{1}{21}+\frac{1}{27}-\frac{1}{33}+\frac{1}{39}- \text{etc.}$$

quae vtique continet omnes numeros compoſitos per 3 diuiſibiles, vnde ſi haec ſeries ad illam addatur, omnes iſti numeri compoſiti excludentur ſietque

$$\frac{4}{3}A-\frac{1}{3} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}+\frac{1}{25} \text{ etc.} = B,$$

ita vt ſit $B = \frac{4}{3}A-\frac{1}{3}$. In hac igitur ſerie, cuius ſumma nouimus, nulli amplius numeri compoſiti per 3 diuiſibiles occurrunt, ſed primus terminus compoſitus hic occurrens eſt $+\frac{1}{25}$.

§. 6. Nunc igitur ex poſtrema hac ſerie omnes terminos per 5 diuiſibiles excludamus, quem in finem hinc formemus iſtam ſeriem:

$$(B-1+\tfrac{1}{3})\tfrac{1}{5} = \tfrac{1}{25}-\tfrac{1}{35}-\tfrac{1}{55}+\tfrac{1}{65}+\tfrac{1}{85}-\tfrac{1}{95}- \text{etc.},$$

quae omnes complectitur terminos per quinque diuiſibiles, qui adhuc in ſeriem B ingrediebantur, er quidem iisdem ſignis affectos, quare hac ſerie ab illa ablata remanebit haec:

$$\tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3}) = 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23}+ \text{etc.}$$

vbi iam ab initio omnes termini ſunt numeri primi, et non primus qui hic occurret erit $\frac{1}{49}$, ſequentes vero $\frac{1}{77}, -\frac{1}{91}$. Ponamus autem ſummam huius ſeriei $= C$, vt ſit

$$C = \tfrac{4}{5}B+\tfrac{1}{5}(1-\tfrac{1}{3}).$$

§. 7. Nunc igitur hinc excludamus omnes terminos qui adhuc per ſeptem ſunt diuiſibiles, quos complectetur ſequens forma:

$$\left(C - 1 + \tfrac{1}{3} - \tfrac{1}{5}\right)\tfrac{1}{7} = -\tfrac{1}{49} - \tfrac{1}{77} + \tfrac{1}{91} + \text{etc.}$$

quorum terminorum ſigna ſunt contraria, quare haec series ad ſeriem C addita dabit

$$\tfrac{8}{7}C - \tfrac{1}{7}\left(1 - \tfrac{1}{3} + \tfrac{1}{5}\right) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} + \text{etc.}$$

in qua primus terminus compoſitus erit $+\frac{1}{121}$; ſequentes vero $-\frac{1}{11.13}$, $-\frac{1}{11.17}$ etc. Hanc autem ſeriem vocemus D vt ſit

$$D = \tfrac{8}{7}C - \tfrac{1}{7}\left(1 - \tfrac{1}{3} + \tfrac{1}{5}\right).$$

§. 8. Iam ex ſerie modo inuenta D expungamus terminos, qui adhuc ſunt per 11 diuiſibiles, quos complectetur iſta forma:

$$D - \left(1 + \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7}\right)\tfrac{1}{11} = -\tfrac{1}{121} + \tfrac{1}{143} + \tfrac{1}{187}\ \text{etc.}$$

qui termini in ſerie D contraria habent ſigna; quamobrem ſi haec ſeries ad illam addatur, iſti termini excludentur, prodibitque

$$\tfrac{12}{11}D - \tfrac{1}{11}\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}\right) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \text{etc.}$$

in qua primus terminus non primus eſt $\frac{1}{169}$; iſtam autem ſeriem deſignemus littera E, ita vt ſit

$$E = \tfrac{12}{11}D - \tfrac{1}{11}\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7}\right).$$

§. 9. Ex hac igitur ſerie excludamus omnes terminos, qui adhuc inſunt per 13 diuiſibiles, quos ergo complectetur haec forma:

$$\left(E - 1 + \tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11}\right)\tfrac{1}{13} = \tfrac{1}{169} + \tfrac{1}{221}\ \text{etc.}$$

hique termini eadem habent ſigna ac in ipſa ſerie E. Haec igitur ſeries ab illa debet ſubtrahi, vnde prodit

$$\tfrac{12}{13}E + \tfrac{1}{13}\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11}\right) = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \text{etc.}$$

vbi

vbi primus terminus non primus est $\frac{1}{289}$. Totam autem hanc feriem designemus littera F, vt sit

$$F = \tfrac{12}{13}E + \tfrac{1}{13}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11}).$$

§. 10. Quod si nunc istas operationes vlterius continuemus, dum successiue hinc excludimus terminos adhuc per 17 diuisibiles, tum vero per 19, per 23, etc. tandem relinquetur tantum series numerorum primorum post vnitatem sequentium, quae si designetur littera Z, quam vt infinitesimam spectari oportet, erit vtique

$$Z = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{19} \text{ etc.}$$

consequenter summa seriei in titulo propositae summa erit $1 - Z$. Ac manifestum est, ad hunc valorem continuo propius accedere istas formulas:

$$1-A,\ 1-B,\ 1-C,\ 1-D,\ 1-E,\ 1-F, \text{ etc.}$$

§. 11. Quemadmodum autem valores omnium harum litterarum successiue ex antecedentibus colligi debeant, ex sequentibus formulis fiet manifestum:

$$B = \tfrac{4}{3}A - \tfrac{1}{3}.1$$
$$C = \tfrac{4}{5}B + \tfrac{1}{5}(1 - \tfrac{1}{3})$$
$$D = \tfrac{8}{7}C - \tfrac{1}{7}(1 - \tfrac{1}{3} + \tfrac{1}{5})$$
$$E = \tfrac{12}{11}D - \tfrac{1}{11}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7})$$
$$F = \tfrac{12}{13}E + \tfrac{1}{13}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11})$$
$$G = \tfrac{16}{17}F + \tfrac{1}{17}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13})$$
$$H = \tfrac{20}{19}G - \tfrac{1}{19}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17})$$
$$I = \tfrac{24}{23}H - \tfrac{1}{23}(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} + \tfrac{1}{17} - \tfrac{1}{19})$$

etc. etc.

 Vbi

Vbi notandum, ſi denominator primus fuerit formae $4n+1$, tum numeratorem primae partis fore vnitate minorem, ſiue $4n$, alteram vero partem addi debere. Sin autem denominator primus fuerit $4n-1$, tum numeratorem primae partis fore vnitate maiorem, ſiue $4n$, alteram partem vero hoc caſu ſubtrahi debere.

§. 12. Quo nunc omnes hos valores in numeris per fractiones decimales exprimamus, ante omnia notetur eſſe

$$A = \frac{\pi}{4} = 0,7853981634.$$

Pro reliquis autem litteris computentur ſequentes valores:

$1-\frac{1}{3}=$ - - - - - - - - - - $b=0,6666666666$

$1-\frac{1}{3}+\frac{1}{5}=$ - - - - - - - - - $c=0,8666666666$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}=$ - - - - - - - - $d=0,7238095238$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}=$ - - - - - - $e=0,6329004329$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}=$ - - - - - $f=0,7098235098$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}=$ - - - $g=0,7686470392$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}=$ - - $h=0,7160154603$

$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}=$ - $i=0,6725371994$

in quo ordine primus terminus a vnitate aequatur.

§. 13. Pro computo autem ipſarum litterarum A, B, C, D, E, etc. praeſtabit ſequentibus vti formulis, quibus ſimul valores numericos harum litterarum adſcribamus

$$B = A + \frac{1}{3}(A-a) = 0,713864$$

$$C = B - \frac{1}{5}(B-b) = 0,704424$$

$$D = C + \frac{1}{7}(C-c) = 0,681247$$

E=

$$E = D + \tfrac{1}{11}(D - d) = 0,677377$$
$$F = E - \tfrac{1}{13}(E - e) = 0,673956$$
$$G = F - \tfrac{1}{17}(F - f) = 0,676066$$
$$H = G + \tfrac{1}{19}(G - g) = 0,671193$$
$$I = H + \tfrac{1}{23}(H - h) = 0,669245$$
$$K = I - \tfrac{1}{29}(I - i) = 0,669358.$$

§. 14. Quanquam autem calculum huc vsque produximus, tamen vltra tertiam figuram decimalem de ſumma noſtrae ſeriei certi eſſe non poſſumus, atque adeo in dubio relinquere cogimur, vtrum iſta ſumma aliquanto maior vel minor ſit quam 0, 669. Sin autem hunc valorem pro vero aſſumamus, ipſa ſeries propoſita

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} \text{ etc.}$$

ſummam habebit 0, 331, ideoque hic valor tantillo foret minor quam $\frac{1}{3}$. Quoniam vero ablato $\frac{1}{5}$, iterum addi debeat $\frac{1}{7} + \frac{1}{11}$, quarum fractionum ſumma maior eſt quam $\frac{1}{5}$, vtique fieri poſſet, vt verus valor ſuperaret $\frac{1}{3}$, id quod hoc loco in dubio eſt relinquendum. Datur vero alia methodus, multo accuratius in ſummam huius ſeriei inquirendi, quam hic euoluemus, quandoquidem operae pretium videtur, veram huius ſeriei ſummam propius cognouiſſe.

§. 15. Eadem methodo qua hic ex prima ſerie Leibnitziana ſucceſſiue terminos compoſitos expulimus, ſi omnes plane terminos praeter vnitatem remoueamus, reperiemus

$$\tfrac{\pi}{4} = \tfrac{3}{4} \cdot \tfrac{5}{4} \cdot \tfrac{7}{8} \cdot \tfrac{11}{12} \cdot \tfrac{13}{12} \cdot \tfrac{17}{16} \cdot \tfrac{19}{20} \cdot \text{ etc.}$$

vbi

vbi in numeratoribus omnes numeri primi occurrunt praeter 2, denominatores vero ſunt numeri pariter pares vnitate vel maiores vel minores. Deinde vero ſi iſta ſeries reciproca quadratorum imparium:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

cuius ſummam oſtendi eſſe $= \frac{\pi\pi}{8}$, ſimili modo tractetur, reperietur

$$\frac{\pi\pi}{8} = \frac{3.\,3}{2.\,4} \cdot \frac{5.\,5}{4.\,6} \cdot \frac{7.\,7}{6.\,8} \cdot \frac{11.\,11}{10.\,12} \cdot \frac{13.\,13}{12.\,14} \cdot \text{etc.}$$

vbi iterum in numeratoribus omnes numeri primi bis occurrunt, in denominatoribus vero iidem tam vnitate aucti quam minuti. Quare ſi hanc expreſſionem per quadratum illius, quod eſt

$$\frac{\pi\pi}{16} = \frac{3.\,3}{4.\,4} \cdot \frac{5.\,5}{4.\,4} \cdot \frac{7.\,7}{8.\,8} \cdot \frac{11.\,11}{12.\,12} \cdot \frac{13.\,13}{12.\,12} \text{ etc.}$$

diuidamus, quotus erit

$$2 = \frac{4}{2} \cdot \frac{4}{6} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{12}{14} \text{ etc.}$$

vbi omnes numeri primi tam vnitate aucti quam minuti occurrunt, et numeri pariter pares in numeratore, impariter pares vero in denominatore conſtituuntur.

§. 16. Poſtrema haec expreſſio igitur hoc modo exhiberi poterit:

$$2 = \frac{3+1}{3-1} \cdot \frac{5-1}{5+1} \cdot \frac{7+1}{7-1} \cdot \frac{11+1}{11-1} \cdot \frac{13-1}{13+1} \text{ etc.}$$

hinc ergo logarithmis hyperbolicis ſumendis habebimus

$$l\,2 = l\frac{3+1}{3-1} + l\frac{5-1}{5+1} + l\frac{7+1}{7-1} + l\frac{11+1}{11-1} + l\frac{13-1}{13+1} + \text{etc.}$$

Conſtat autem per ſeries infinitas eſſe in genere

$$\frac{1}{2}\,l\frac{a+1}{a-1} = \frac{1}{a} + \frac{1}{3a^3} + \frac{1}{5a^5} + \frac{1}{7a^7} + \frac{1}{9a^9} + \text{etc.}$$

hincque

hincque

$$\tfrac{1}{2} l \frac{a-1}{a+1} = -\frac{1}{a} - \frac{1}{3a^3} - \frac{1}{5a^5} - \frac{1}{7a^7} - \frac{1}{9a^9} - \text{etc.}$$

Quodsi igitur harum formularum ope omnes illos logarithmos in series infinitas conuertamus, nanciscemur quidem innumeras series infinitas, quas autem ad series facilius tractabiles reducere licebit.

§. 17. Primo igitur omnium illorum logarithmorum semisses accipi oportet, et quia hic de logarithmis hyperbolicis agitur, ob

$$l\,2 = 0,6931471805, \text{ erit}$$

$$\tfrac{1}{2} l\,2 = 0,3465735902,$$

ex altera autem parte logarithmi ita ordinentur:

$$\tfrac{1}{2} l \frac{3+1}{3-1} = \frac{1}{3} + \frac{1}{3.\,3^3} + \frac{1}{5.\,3^5} + \frac{1}{7.\,3^7} + \frac{1}{9.\,3^9} + \text{etc.}$$

$$\tfrac{1}{2} l \frac{5-1}{5+1} = -\frac{1}{5} - \frac{1}{3.\,5^3} - \frac{1}{5.\,5^5} - \frac{1}{7.\,5^7} - \frac{1}{9.\,5^9} - \text{etc.}$$

$$\tfrac{1}{2} l \frac{7+1}{7-1} = \frac{1}{7} + \frac{1}{3.\,7^3} + \frac{1}{5.\,7^5} + \frac{1}{7.\,7^7} + \frac{1}{9.\,7^9} + \text{etc.}$$

$$\tfrac{1}{2} l \frac{11+1}{11-1} = \frac{1}{11} + \frac{1}{3.\,11^3} + \frac{1}{5.\,11^5} + \frac{1}{7.\,11^7} + \frac{1}{9.\,11^9} + \text{etc.}$$

$$\tfrac{1}{2} l \frac{13-1}{13+1} = -\frac{1}{13} - \frac{1}{3.\,13^3} - \frac{1}{5.\,13^5} - \frac{1}{7.\,13^7} - \frac{1}{9.\,13^9} - \text{etc.}$$

etc. etc.

§. 18. Hinc iam verticaliter descendendo consideremus sequentes series pariter infinitas:

$$O = \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{17} + \frac{1}{19} + \text{etc.}$$

$$P = \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} + \text{etc.}$$

$$Q = \frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} - \frac{1}{13^5} - \frac{1}{17^5} + \frac{1}{19^5} + \text{etc.}$$

$$R = \frac{1}{3^7} - \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} - \frac{1}{13^7} - \frac{1}{17^7} + \frac{1}{19^7} + \text{etc.}$$

$$S = \frac{1}{3^9} - \frac{1}{5^9} + \frac{1}{7^9} + \frac{1}{11^9} - \frac{1}{13^9} - \frac{1}{17^9} + \frac{1}{19^9} + \text{etc.}$$

etc. etc.

Quarum ſerierum prima O eſt ea ipſa, cuius ſummam hic inueſtigare nobis eſt propoſitum.

§. 19. His igitur ſeriebus ita per litteras maiusculas deſignatis habebimus iſtam aequationem:

$$\tfrac{1}{2} l 2 = O + \tfrac{1}{3} P + \tfrac{1}{5} Q + \tfrac{1}{7} R + \tfrac{1}{9} S + \tfrac{1}{11} T + \text{etc.}$$

vnde ſi ſummae ſerierum P, Q, R, S eſſent cognitae, inde impetraremus facile ſummam ſeriei O quaeſitam; foret enim

$$O = \tfrac{1}{2} l 2 - \tfrac{1}{3} P - \tfrac{1}{5} Q - \tfrac{1}{7} R - \tfrac{1}{9} S - \text{etc.}$$

§. 20. At vero ſummas ſerierum P, Q, R etc. ex ſeriebus ordinis, vbi omnes numeri impares occurrunt, concludere poterimus eodem modo quo ſupra ſeriem ipſam O ex ſerie Leibnitziana

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} \text{ etc.}$$

elicuimus. Hunc in finem euolui hac methodo oportebit ſequentes ſeries ordinatas:

$$\mathfrak{P} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + \text{etc.}$$

$$\mathfrak{Q} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \frac{1}{13^5} - \frac{1}{15^5} + \text{etc.}$$

$$\mathfrak{R} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \frac{1}{13^7} - \frac{1}{15^7} + \text{etc.}$$

$$\mathfrak{S} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \frac{1}{11^9} + \frac{1}{13^9} - \frac{1}{15^9} + \text{etc.}$$

$$\mathfrak{T} = 1 - \frac{1}{3^{11}} + \frac{1}{5^{11}} - \frac{1}{7^{11}} + \frac{1}{9^{11}} - \frac{1}{11^{11}} + \frac{1}{13^{11}} - \frac{1}{15^{11}} + \text{etc.}$$

etc. etc.

harum

Harum autem omnium serierum summas iam pridem per quadraturam circuli, scilicet per similes potestates ipsius π expressas dedi, sequenti modo:

$$\mathfrak{P} = \frac{1}{1.\,2} \cdot \frac{\pi^3}{2^4} \qquad \mathfrak{T} = \frac{50521}{1 \ldots\ldots 10} \cdot \frac{\pi^{11}}{2^{12}}$$

$$\mathfrak{Q} = \frac{5}{1.\,2.\,3.\,4} \cdot \frac{\pi^5}{2^6} \qquad \mathfrak{U} = \frac{2702765}{1 \ldots\ldots 12} \cdot \frac{\pi^{13}}{2^{14}}$$

$$\mathfrak{R} = \frac{61}{1 \ldots\ldots 6} \cdot \frac{\pi^7}{2^8} \qquad \mathfrak{V} = \frac{199360981}{1 \ldots\ldots 14} \cdot \frac{\pi^{15}}{2^{16}}$$

$$\mathfrak{S} = \frac{1385}{1 \ldots\ldots 8} \cdot \frac{\pi^9}{2^{10}} \qquad \mathfrak{W} = \frac{19391512145}{1 \ldots\ldots 16} \cdot \frac{\pi^{17}}{2^{18}}$$

etc. etc.

§. 21. Hos igitur valores in fractionibus decimalibus vsque ad sextam figuram euoluamus, eritque

	Differentiae
$\mathfrak{P} = 0,9689462$	$0,0272095$
$\mathfrak{Q} = 0,9961557$	$0,0033990$
$\mathfrak{R} = 0,9995547$	$0,0003952$
$\mathfrak{S} = 0,9999499$	$0,0000448$
$\mathfrak{T} = 0,9999947$	$0,0000050$
$\mathfrak{U} = 0,9999997$	$0,0000005$
etc.	etc.

§. 22. Vt nunc hinc valores litterarum P, Q, R, etc. eruamus, eadem methodo vtamur, qua supra ex serie

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}$$

omnes terminos compositos exterminauimus, quandoquidem loco horum numerorum simplicium eorum potestates scribi conuenit. Hanc igitur operationem in genere pro his litteris doceamus. Consideremus igitur hanc seriem:

$$\mathfrak{Z} = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \text{etc.}$$

cuius summam, vt supra factum est, littera A designemus; vt sit $A = \mathfrak{Z}$, hincque sequentes litteras B, C, D etc. eliciamus per sequentes formulas:

$$B = A + \frac{1}{3^n}(A - a) \text{ existente } a = 1$$

$$C = B - \frac{1}{5^n}(B - b) \text{ - - - - } b = 1 - \frac{1}{3^n}$$

$$D = C + \frac{1}{7^n}(C - c) \text{ - - - - } c = 1 - \frac{1}{3^n} + \frac{1}{5^n}$$

$$E = D + \frac{1}{11^n}(D - d) \text{ - - - - } d = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n}$$

$$F = E - \frac{1}{13^n}(E - e) \text{ - - - - } e = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n}$$

etc. etc.

Quibus valoribus inuentis eorum complementa ad vnitatem, scilicet: $1 - A$, $1 - B$, $1 - C$, $1 - D$, etc. promptissime ad valorem quaesitum

$$Z = \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} - \frac{1}{13^n} - \text{etc.}$$

appropinquabunt.

§. 23. Haec igitur praecepta generalia applicemus primo ad valorem litterae P, vnde incipiendum erit a valore

$$\mathfrak{P} = 0,9689462 = A,$$

et quia hic est $n = 3$, habebimus

$$a = 1, \; b = 0,9629630, \; c = 0,9709630, \; d = 0,9680476;$$

pluri-

pluribus valoribus non erit opus. Hinc igitur colligemus sequentes valores:

$$B = A - \tfrac{1}{3^3} . 0,0310538 = 0,9677961$$

$$C = B - \tfrac{1}{5^3} . 0,0048331 = 0,9677574$$

$$D = C - \tfrac{1}{7^3} . 0,0032056 = 0,9677481$$

$$E = D - \tfrac{1}{11^3} . 0,0002995 = 0,9677479$$

Vlterius procedi non est opus; quamobrem hinc habebimus

$$P = 1 - E = 0,0322521,$$

vnde iam colligimus

$$\tfrac{1}{2} l 2 - \tfrac{1}{3} . 0,0322521 = 0,3358229.$$

§. 24. Sumamus nunc $n = 5$ et habebimus

$$A = \mathfrak{Q} = 0,9961557,$$

tum vero erit

$a = 1$; $b = 0,9958847$; $c = 0,9962048$; $d = 0,9961753$;

hinc igitur reperiemus

$$B = A - \tfrac{1}{3^5} . 0,0038443 = 0,9961899$$

$$C = B - \tfrac{1}{5^5} . 0,0002551 = 0,9961398.$$

Erit igitur

$$Q = 1 - C = 0,0038602, \text{ ideoque}$$

$$\tfrac{1}{2} l 2 - \tfrac{1}{3} P - \tfrac{1}{3} Q = 0,3350509.$$

§. 25. Sit nunc $n = 7$ et $A = \mathfrak{R} = 0,9995547$, tum vero $a = 1$; $b = 0,9995428$, hinc igitur fiet

$$B = A - \tfrac{1}{3^7} 0,0004453 = 0,9995545,$$

 vnde

vnde iam habemus

$$R = 1 - B = 0,0004455 \text{ ideoque}$$

$$\tfrac{1}{2} l 2 - \tfrac{1}{3} P - \tfrac{1}{5} Q - \tfrac{1}{7} R = 0,3349873.$$

§. 26. Cum in hoc calculo tantum non fuerit $B = A$, in sequentibus nequidem littera B erit opus, quamobrem habebimus

$$S = 1 - \mathfrak{S} = 0,0000501 \text{ hincque}$$

$$\tfrac{1}{9} S = 0,0000056.$$

Deinde vero erit

$$T = 1 - \mathfrak{T} = 0,0000053 \text{ hincque}$$

$$\tfrac{1}{11} T = 0,0000005, \text{ denique}$$

$$U = 1 - \mathfrak{U} = 0,0000003 \text{ et } \tfrac{1}{13} U = 0,000000.$$

Particulis igitur his a praecedente valore ablatis prodit

$$O = 0,3349812.$$

Vnde patet, hunc valorem adhuc aliquanto maiorem esse quam $\frac{1}{3}$.

§. 27. Nunc igitur certi esse possumus summam seriei infinitae huius

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \tfrac{1}{19} + \text{ etc.}$$

esse satis exacte $= 0,3349812$. Inuestigandum iam foret, num iste valor non quampiam teneat rationem notabilem, siue ad peripheriam circuli π, siue ad eius logarithmum hyperbolicum, quandoquidem supra obseruauimus, seriem reciprocam numerorum primorum

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} + \text{ etc.}$$

expri-

exprimere logarithmum hyperbolicum seriei harmonicae completae

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \text{etc.}$$

vnde videri potest, istam seriem numerorum primorum

$$\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{13} - \tfrac{1}{17} + \text{etc.}$$

etiam continere logarithmum eiusdem seriei completae

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \text{etc.}$$

cuius summa est $\frac{\pi}{4}$. Hunc in finem subiungam logarithmum hyperbolicum ipsius π, quem olim reperi

$$1,14472,98858,49400,17414,34273,51353,05865.$$

Videndum igitur erit num forte sit summa inuenta $O = l\pi - lN$, ita vt N sit numerus satis simplex. Verum huiusmodi inuestigationes plerumque sine vllo successu instituuntur.

§. 28. Ope posterioris methodi autem non solum summam seriei propositae elicuimus, sed etiam eius potestatum imparium, quas summas hic conspectui exponamus.

$$\frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{17} + \frac{1}{19} + \text{etc.} = 0,3349812$$
$$\frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} + \text{etc.} = 0,0322521$$
$$\frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} - \frac{1}{13^5} - \frac{1}{17^5} + \frac{1}{19^5} + \text{etc.} = 0,0038602$$
$$\frac{1}{3^7} - \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} - \frac{1}{13^7} - \frac{1}{17^7} + \frac{1}{19^7} + \text{etc.} = 0,0004455$$
$$\frac{1}{3^9} - \frac{1}{5^9} + \frac{1}{7^9} + \frac{1}{11^9} - \frac{1}{13^9} - \frac{1}{17^9} + \frac{1}{19^9} + \text{etc.} = 0,0000501$$
$$\frac{1}{3^{11}} - \frac{1}{5^{11}} + \frac{1}{7^{11}} + \frac{1}{11^{11}} - \frac{1}{23^{11}} - \frac{1}{17^{11}} + \frac{1}{19^{11}} + \text{etc.} = 0,0000056$$
$$\frac{1}{3^{13}} - \frac{1}{5^{13}} + \frac{1}{7^{13}} + \frac{1}{11^{13}} - \frac{1}{13^{13}} - \frac{1}{17^{13}} + \frac{1}{19^{13}} + \text{etc.} = 0,0000005$$

etc. etc.

§. 29.

§. 29. Ipſae quidem hae ſummae ſine dubio parum attentionis merentur, niſi forte ad quantitates cognitas reduci potuerint. Verum quia in his ſeriebus neque ipſi termini ſecundum certam legem progrediuntur, neque etiam in ſignis plus vel minus certus ordo obſeruatur; iſta disquiſitio primo intuitu plane impoſſibilis videri potuiſſet, quamobrem ipſa methodus, qua ad earum ſummas pertigimus, vtique omni attentione digna eſt cenſenda, idque eo magis, quod ſatis abſtruſis ſerierum poteſtatum proprietatibus innititur. Niſi enim ſummae ſerierum

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} \text{ etc.}$$

pro caſibus quibus n eſt numerus impar, fuiſſent cognitae, tota haec inueſtigatio fruſtra fuiſſet ſuſcepta.

DE

DE SERIEBVS POTESTATVM RECIPROCIS METHODO NOVA ET FACILLIMA SVMMANDIS.

§ 1.

Cum primum summas harum serierum docuissem, eas ex hoc principio deduxi, quod cuique sinui et cosinui innumerabiles arcus circulares respondent, qui omnes sint radices aequationum infinitarum, quibus arcus per sinum vel cosinum exprimi solent. Hinc enim ex coefficientibus istarum aequationum non solum summas ipsarum radicum, sed etiam earum potestatum quarumcunque assignaui. Postea vero easdem summas etiam ex aliis principiis deriuaui, quae autem omnia memorata circuli proprietate innitebantur. Nunc vero obseruaui, istas summas ex alio principio multo simpliciori, et solis operationibus analyticis innixo, deduci posse, quam methodum hic accuratius exposuisse iuuabit.

§. 2. Hoc autem principium mihi suppeditauit integratio huius formulae: $\int\left(\frac{z^{m-1}+z^{n-m-1}}{1+z^n}\right)dz$, pro casu quo post integrationem statuitur $z=1$. Ostendi enim in

Tomo XIX Nov. Comment. per ſolitas integrationum operationes haec integralia ſequenti modo exprimi:

$$\int\left(\frac{z^{m-1}+z^{n-m-1}}{1+z^n}\right)dz=\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}}\ \text{et}$$

$$\int\left(\frac{z^{m-1}-z^{n-m-1}}{1-z^n}\right)dz=\frac{\pi}{n\,\text{tang.}\,\frac{m\pi}{n}}.$$

Quod ſi vero eaedem formulae per ſeries infinitas euoluantur, poſito $z=1$ erit

$$\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}}=\frac{1}{m}-\frac{1}{n+m}+\frac{1}{2n+m}-\frac{1}{3n+m}+\frac{1}{4n+m}-\text{etc.}$$
$$+\frac{1}{n-m}-\frac{1}{2n-m}+\frac{1}{3n-m}-\frac{1}{4n-m}+\text{etc.}\ \text{et}$$

$$\frac{\pi}{n\,\text{tang.}\,\frac{m\pi}{n}}=\frac{1}{m}+\frac{1}{n+m}+\frac{1}{2n+m}+\frac{1}{3n+m}+\text{etc.}$$
$$-\frac{1}{n-m}-\frac{1}{2n-m}-\frac{1}{3n-m}-\frac{1}{4n-m}-\text{etc.}$$

quae duae ſeries eo maiori attentione ſunt dignae, quod in iis omnia plane continentur, quae non ſolum circa ſummationes poteſtatum, ſed etiam circa ſummationes ſimiles ſunt prolata.

Euolutio prioris ſeriei generalis.

§. 3. Conſideremus primo formam priorem $\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}}$ ac binis terminis analogis contractis habebimus

$$\frac{\pi}{n\,\text{ſin.}\,\frac{m\pi}{n}}=\frac{1}{m}+\frac{2m}{nn-mm}-\frac{2m}{4nn-mm}+\frac{2m}{9nn-mm}-\frac{2m}{16nn-mm}+\text{etc.}$$

Suma-

Sumatur nunc, quo formulae fiant simpliciores, $m = 1$ eritque

$$\frac{\pi}{n \sin. \frac{\pi}{n}} = 1 + \frac{2}{nn-1} - \frac{2}{4nn-1} + \frac{2}{9nn-1} - \frac{2}{16nn-1} + \text{etc.}$$

siue

$$\frac{\pi}{2n \sin. \frac{\pi}{n}} - \frac{1}{2} = \frac{1}{nn-1} - \frac{1}{4nn-1} + \frac{1}{9nn-1} - \frac{1}{16nn-1} + \text{etc.}$$

§. 4. Nunc singulas has fractiones more solito in series infinitas geometricas resoluamus eritque

$$\begin{aligned}
\frac{1}{nn-1} &= \frac{1}{nn} + \frac{1}{n^4} + \frac{1}{n^6} + \frac{1}{n^8} + \frac{1}{n^{10}} + \text{etc.} \\
-\frac{1}{4nn-1} &= -\frac{1}{4nn} - \frac{1}{4^2 n^4} - \frac{1}{4^3 n^6} - \frac{1}{4^4 n^8} - \frac{1}{4^5 n^{10}} - \text{etc.} \\
+\frac{1}{9nn-1} &= \frac{1}{9nn} + \frac{1}{9^2 n^4} + \frac{1}{9^3 n^6} + \frac{1}{9^4 n^8} + \frac{1}{9^5 n^{10}} + \text{etc.} \\
-\frac{1}{16nn-1} &= -\frac{1}{16nn} - \frac{1}{16^2 n^4} - \frac{1}{16^3 n^6} - \frac{1}{16^4 n^8} - \frac{1}{16^5 n^{10}} - \text{etc.} \\
\text{etc.} & \qquad \text{etc.}
\end{aligned}$$

Harum igitur serierum infinitarum omnium summa erit

$$= \frac{\pi}{2n \sin. \frac{\pi}{n}} - \frac{1}{2}.$$

§. 5. Nunc igitur has series secundum lineas verticales colligamus, quem in finem statuamus breuitatis gratia

$$\begin{aligned}
1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} &= A\pi\pi \\
1 - \frac{1}{4^2} + \frac{1}{9^2} - \frac{1}{16^2} + \frac{1}{25^2} - \text{etc.} &= B\pi^4 \\
1 - \frac{1}{4^3} + \frac{1}{9^3} - \frac{1}{16^3} + \frac{1}{25^3} - \text{etc.} &= C\pi^6 \\
1 - \frac{1}{4^4} + \frac{1}{9^4} - \frac{1}{16^4} + \frac{1}{25^4} - \text{etc.} &= D\pi^8 \\
\text{etc.} \qquad & \text{etc.}
\end{aligned}$$

hinc igitur adipiſcemur ſequentem aequationem:

$$\frac{\pi}{2n \text{ ſin.} \frac{\pi}{n}} - \frac{1}{2} = \frac{A\pi\pi}{nn} + \frac{B\pi^4}{n^4} + \frac{C\pi^6}{n^6} + \frac{D\pi^8}{n^8} + \text{etc.}$$

§. 6. Ponamus porro breuitatis gratia $\frac{\pi}{n} = x$, vt prodeat ſequens aequatio:

$$\frac{x}{2 \text{ ſin. } x} - \frac{1}{2} = Axx + Bx^4 + Cx^6 + Dx^8 + Ex^{10} + \text{etc.}$$

vbi iam intelligitur, per debitam euolutionem omnes coëfficientes aſſumtos A, B, C, etc. definiri poſſe, quibus inventis nanciſcemur ſummas omnium ſerierum in hac forma contentarum:

$$1 - \frac{1}{4^i} + \frac{1}{9^i} - \frac{1}{16^i} + \frac{1}{25^i} - \text{etc.}$$

ſiue in hac:

$$1 - \frac{1}{2^{2i}} + \frac{1}{3^{2i}} - \frac{1}{4^{2i}} + \frac{1}{5^{2i}} - \text{etc.}$$

denotante i numerum integrum quemcunque.

§. 7. Cum iam per ſeriem notiſſimam ſit

$$\text{ſin. } x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1 \ldots\ldots 5} - \frac{x^7}{1 \ldots\ldots 7} + \text{etc.}$$

pro hac ſerie ſimpliciter ſcribamus

$$\text{ſin. } x = \alpha x - \beta x^3 + \gamma x^5 - \delta x^7 + \varepsilon x^9 - \text{etc.}$$

ita vt ſit

$$\alpha = 1, \ \beta = \frac{\alpha}{2.3}, \ \gamma = \frac{\beta}{4.5}, \ \delta = \frac{\gamma}{6.7}, \ \varepsilon = \frac{\delta}{8.9}, \text{ etc.}$$

quo poſito membrum $-\frac{1}{2}$ ad dextram partem transferamus atque vtrinque multiplicemus per hanc ſeriem ipſi ſin x aequalem, fietque

$\frac{x}{2} = \frac{1}{2}$

$$
\begin{array}{lllllllll}
\frac{x}{2} = \frac{1}{2}\alpha x & + \alpha A x^3 & + \alpha B x^5 & + \alpha C x^7 & + \alpha D x^9 & + \alpha E x^{11} & + \alpha F x^{13} \text{ etc.} \\
 & - \frac{1}{2}\beta & - \beta A & - \beta B & - \beta C & - \beta D & - \beta E \\
 & & + \frac{1}{2}\gamma & + \gamma A & + \gamma B & + \gamma C & + \gamma D \\
 & & & - \frac{1}{2}\delta & - \delta A & - \delta B & - \delta C \\
 & & & & + \frac{1}{2}\varepsilon & + \varepsilon A & + \varepsilon B \\
 & & & & & - \frac{1}{2}\zeta & - \zeta A \\
 & & & & & & - \frac{1}{2}\eta.
\end{array}
$$

§. 8. Quoniam haec aequalitas ſubſiſtere debet; quicunque valor litterae x tribuatur, ſingulae eius poteſtates ſe mutuo ſeorſim deſtruere debent. Primo quidem termini ipſum x continentes ob $\alpha = 1$ ſponte ſe tollunt, reliquae poteſtates ob $\alpha = 1$ ſequentes dant determinationes:

$$
\begin{aligned}
A &= \tfrac{1}{2}\beta \\
B &= \beta A - \tfrac{1}{2}\gamma \\
C &= \beta B - \gamma A + \tfrac{1}{2}\delta \\
D &= \beta C - \gamma B + \delta A - \tfrac{1}{2}\varepsilon \\
E &= \beta D - \gamma C + \delta B - \varepsilon A + \tfrac{1}{2}\zeta \\
&\text{etc.} \qquad \text{etc.}
\end{aligned}
$$

Harum igitur formularum ope ſummae quantumuis altarum poteſtatum parium aſſignari poterunt.

§. 9. Inuenta autem ſumma huius ſeriei:

$$s = 1 - \frac{1}{2^{2i}} + \frac{1}{3^{2i}} - \frac{1}{4^{2i}} + \frac{1}{5^{2i}} - \text{etc.}$$

ex ea quoque ſerierum agnatarum iſtarum ſummae definiri poterunt:

$$t = 1 + \frac{1}{3^{2i}} + \frac{1}{5^{2i}} + \frac{1}{7^{2i}} + \frac{1}{9^{2i}} + \text{etc. et}$$

$$u = 1 + \frac{1}{2^{2i}} + \frac{1}{3^{2i}} + \frac{1}{4^{2i}} + \frac{1}{5^{2i}} + \frac{1}{6^{2i}} + \text{etc.}$$

Cum enim ſit

$t =$

$$t = u\left(1 - \frac{1}{2^{2i}}\right) = \left(\frac{2^{2i}-1}{2^{2i}}\right)u \text{ et}$$

$$s = u\left(1 - \frac{2}{2^{2i}}\right) = \left(\frac{2^{2i}-2}{2^{2i}}\right)u, \text{ erit}$$

$$u = \frac{2^{2i}s}{2^{2i}-2}, \text{ hincque } t = \left(\frac{2^{2i}-1}{2^{2i}-2}\right)s$$

in ſequentibus autem harum ſerierum ſummae etiam immediate ex noſtris formulis generalibus elicientur.

Euolutio ſeriei generalis poſterioris.

§. 10. Quod ſi hic etiam bini termini analogi contrahantur, orietur iſta ſeries:

$$\frac{\pi}{n \text{ tang.} \frac{m\pi}{n}} = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm}$$
$$- \frac{2m}{16nn-mm} + \text{etc.}$$

Ponamus hic iterum $m = 1$, et facta diuiſione per 2 habebimus

$$\frac{1}{nn-1} + \frac{1}{4nn-1} + \frac{1}{9nn-1} + \frac{1}{16nn-1} + \frac{1}{25nn-1} + \text{etc.}$$
$$= \frac{1}{2} - \frac{\pi}{2n \text{ tang.} \frac{\pi}{n}}.$$

Nunc ſingulae iſtae fractiones in ſeries reſoluantur vt ſupra, eritque

$$\frac{1}{nn-1} = \frac{1}{nn} + \frac{1}{n^4} + \frac{1}{n^6} + \frac{1}{n^8} + \frac{1}{n^{10}} + \frac{1}{n^{12}} + \text{etc.}$$
$$\frac{1}{4nn-1} = \frac{1}{4nn} + \frac{1}{4^2 n^4} + \frac{1}{4^3 n^6} + \frac{1}{4^4 n^8} + \frac{1}{4^5 n^{10}} + \frac{1}{4^6 n^{12}} + \text{etc.}$$
$$\frac{1}{9nn-1} = \frac{1}{9nn} + \frac{1}{9^2 n^4} + \frac{1}{9^3 n^6} + \frac{1}{9^4 n^8} + \frac{1}{9^5 n^{10}} + \frac{1}{9^6 n^{12}} + \text{etc.}$$
$$\frac{1}{16nn-1} = \frac{1}{16nn} + \frac{1}{16^2 n^4} + \frac{1}{16^3 n^6} + \frac{1}{16^4 n^8} + \frac{1}{16^5 n^{10}} + \frac{1}{16^6 n^{12}} + \text{etc.}$$
etc. etc.

cuncta-

cunctarum igitur harum ſerierum iunctim ſumtarum ſumma erit $= \frac{1}{2} - \frac{\pi}{2n \operatorname{tang.} \frac{\pi}{n}}$.

§. 11. Nunc igitur, vt ſupra fecimus, per columnas verticales ſummam colligamus, quem in finem ſtatuamus

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \mathfrak{A}\pi^2$$
$$1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \text{etc.} = \mathfrak{B}\pi^4$$
$$1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \text{etc.} = \mathfrak{C}\pi^6$$
$$1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \text{etc.} = \mathfrak{D}\pi^8$$
etc. etc.

Quibus poſitis aequatio noſtra erit

$$\frac{1}{2} - \frac{\pi}{2n \operatorname{tang.} \frac{\pi}{n}} = \frac{\mathfrak{A}\pi^2}{n^2} + \frac{\mathfrak{B}\pi^4}{n^4} + \frac{\mathfrak{C}\pi^6}{n^6} + \frac{\mathfrak{D}\pi^8}{n^8} + \text{etc.}$$

§. 12. Faciamus nunc $\frac{\pi}{n} = x$, quo pacto ambae litterae π et n ſimul ex calculo elidentur, eritque

$$\frac{1}{2} - \frac{x}{2 \operatorname{tang.} x} = \mathfrak{A}xx + \mathfrak{B}x^4 + \mathfrak{C}x^6 + \mathfrak{D}x^8 + \mathfrak{E}x^{10} + \text{etc.}$$

vbi loco huius ſeriei breuitatis gratia ſcribamus litteram s, vt ſit

$$s = \frac{1}{2} - \frac{x}{2 \operatorname{tang.} x} = \frac{\operatorname{ſin.} x - x \operatorname{coſ.} x}{2 \operatorname{ſin.} x}$$

quae aequatio per ſin. x multiplicata praebet

$$s \operatorname{ſin.} x = \frac{1}{2} \operatorname{ſin.} x - \frac{1}{2} x \operatorname{coſ.} x.$$

§. 13. Statuamus nunc, vt in praecedente euolutione,

$$\operatorname{ſin.} x = \alpha x - \beta x^3 + \gamma x^5 - \delta x^7 + \varepsilon x^9 - \text{etc.}$$

exiſten-

existente

$$\alpha = 1,\ \beta = \frac{1}{1.\,2.\,3}\ \gamma = \frac{1}{1.\,2.\,3.\,4 \ldots 5},\ \delta = \frac{1}{1 \ldots \ldots 7},\ \text{etc.}$$

Quia nunc est

$$\cos x = 1 - \frac{xx}{1.\,2} + \frac{x^4}{1 \ldots 4} - \frac{x^6}{1 \ldots 6} + \frac{x^8}{1 \ldots 8} - \text{etc.}$$

erit

$$\cos x = \alpha - 3\beta xx + 5\gamma x^4 - 7\delta x^6 + 9\varepsilon x^8 - \text{etc.}$$

Tum autem erit

$$\tfrac{1}{2}\sin x - \tfrac{1}{2}x\cos x = \beta x^3 - 2\gamma x^5 + 3\delta x^7 - 4\varepsilon x^9 + 5\zeta x^{11} - \text{etc.}$$

cui ergo expressioni formula $s \sin x$ debet esse aequalis.

§. 14. Binas igitur series per s et $\sin x$ indicatas inuicem multiplicemus, et productum reperietur

$$\begin{array}{rrrrrrl}
s\sin x = \alpha\mathfrak{A}x^3 & + \alpha\mathfrak{B}x^5 & + \alpha\mathfrak{C}x^7 & + \alpha\mathfrak{D}x^9 & + \alpha\mathfrak{E}x^{11} & + \alpha\mathfrak{F}x^{13} & + \text{etc.} \\
& - \beta\mathfrak{A} & - \beta\mathfrak{B} & - \beta\mathfrak{C} & - \beta\mathfrak{D} & - \beta\mathfrak{E} & - \text{etc.} \\
& & + \gamma\mathfrak{A} & + \gamma\mathfrak{B} & + \gamma\mathfrak{C} & + \gamma\mathfrak{D} & + \text{etc.} \\
& & & - \delta\mathfrak{A} & - \delta\mathfrak{B} & - \delta\mathfrak{C} & - \text{etc.} \\
& & & & + \varepsilon\mathfrak{A} & + \varepsilon\mathfrak{B} & + \text{etc.} \\
& & & & & - \zeta\mathfrak{A} & - \text{etc.}
\end{array}$$

quae expressio praecedenti debet esse aequalis.

§. 15. Singulae igitur potestates ipsius x seorsim inter se aequentur, indeque formentur sequentes determinationes:

$\mathfrak{A} = \beta$

$\mathfrak{B} = \beta\mathfrak{A} - 2\gamma$

$\mathfrak{C} = \beta\mathfrak{B} - \gamma\mathfrak{A} + 3\delta$

$\mathfrak{D} = \beta\mathfrak{C} - \gamma\mathfrak{B} + \delta\mathfrak{A} - 4\varepsilon$

$\mathfrak{E} =$

$$\mathfrak{E} = \beta\mathfrak{D} - \gamma\mathfrak{C} + \delta\mathfrak{B} - \varepsilon\mathfrak{A} + 5\zeta$$
$$\mathfrak{F} = \beta\mathfrak{E} - \gamma\mathfrak{D} + \delta\mathfrak{C} - \varepsilon\mathfrak{B} + \zeta\mathfrak{A} - 6\eta$$
etc. etc.

§. 16. Quanquam ope harum formularum determinatio coefficientium $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. quousque libuerit continuari potest, tamen ex iisdem principiis aliae relationes inter hos coefficientes deriuari possunt, quibus calculus haud mediocriter subleuabitur. Resumamus scilicet aequationem $\frac{1}{2} - \frac{x}{2\,\mathrm{tang.}\,x} = s$, vnde fit $\frac{x}{2\,\mathrm{tang.}\,x} = \frac{1}{2} - s$, hincque porro $\frac{1-2s}{x} = \mathrm{cot.}\,x$, quae cotangens statuatur $= t$, vt fit $t = \frac{1-2s}{x}$; ergo loco s serie substituta fiet

$$t = \frac{1}{x} - 2\mathfrak{A}x - 2\mathfrak{B}x^3 - 2\mathfrak{C}x^5 - 2\mathfrak{D}x^7 - \text{etc.}$$

§. 17. Cum igitur posuerimus $\mathrm{cot.}\,x = t$, ideoque $x = \mathrm{A\,cot.}\,t$, erit differentiando $dx = -\frac{dt}{1+tt}$ hincque

$$dt + dx(1+tt) = 0, \text{ siue}$$
$$\frac{dt}{dx} + 1 + tt = 0. \text{ Est vero}$$
$$\frac{dt}{dx} = -\frac{1}{xx} - 2\mathfrak{A} - 6\mathfrak{B}xx - 10\mathfrak{C}x^4 - 14\mathfrak{D}x^6 - 18\mathfrak{E}x^8 - \text{etc.}$$

praeterea vero reperitur

$$\begin{array}{llllllll}
1+tt = \frac{1}{xx} & -4\mathfrak{A} & -4\mathfrak{B}xx & -4\mathfrak{C}x^4 & -4\mathfrak{D}x^6 & -4\mathfrak{E}x^8 & -4\mathfrak{F}x^{10} & -\text{etc.}\\
 & +1 & +4\mathfrak{A}\mathfrak{A} & +8\mathfrak{A}\mathfrak{B} & +8\mathfrak{A}\mathfrak{C} & +8\mathfrak{A}\mathfrak{D} & +8\mathfrak{A}\mathfrak{E} & +\text{etc.}\\
 & & & & +4\mathfrak{B}\mathfrak{B} & +8\mathfrak{B}\mathfrak{C} & +8\mathfrak{B}\mathfrak{D} & +\\
 & & & & & & +4\mathfrak{C}\mathfrak{C} &
\end{array}$$

§. 18. In aequalitate igitur $\frac{dt}{dx} + 1 + tt = 0$, prima membra sponte se tollunt; ex sequentibus autem colliguntur sequentes determinationes:

$\mathfrak{A} = \frac{1}{6}$,

$\mathfrak{B} = \frac{2}{5}\mathfrak{A}\mathfrak{A}$,

$\mathfrak{C} = \frac{2}{7}.2\mathfrak{A}\mathfrak{B}$,

$\mathfrak{D} = \frac{2}{9}(2\mathfrak{A}\mathfrak{C} + \mathfrak{B}\mathfrak{B})$

$\mathfrak{E} = \frac{2}{11}(2\mathfrak{A}\mathfrak{D} + 2\mathfrak{B}\mathfrak{C})$,

$\mathfrak{F} = \frac{2}{13}(2\mathfrak{A}\mathfrak{E} + 2\mathfrak{B}\mathfrak{D} + \mathfrak{C}\mathfrak{C})$

$\mathfrak{G} = \frac{2}{15}(2\mathfrak{A}\mathfrak{F} + 2\mathfrak{B}\mathfrak{E} + 2\mathfrak{C}\mathfrak{D})$,

$\mathfrak{H} = \frac{2}{17}(2\mathfrak{A}\mathfrak{G} + 2\mathfrak{B}\mathfrak{F} + 2\mathfrak{C}\mathfrak{E} + \mathfrak{D}\mathfrak{D})$.

§. 19. Ex his formulis iam olim in introductione mea in Analysin infinitorum valores istarum litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. satis longe computaui, deinceps vero ad aliquot terminos longius continuaui, quos valores igitur hic apponam:

$\mathfrak{A} = \frac{1}{1.2.3} \cdot \frac{1}{1} = \frac{1}{6}$, - - - - - - pro potestatibus secundis

$\mathfrak{B} = \frac{2^2}{1 \ldots 5} \cdot \frac{1}{3} = \frac{1}{90}$, - - - - - pro potestatibus quartis

$\mathfrak{C} = \frac{2^4}{1 \ldots 7} \cdot \frac{1}{3} = \frac{1}{945}$ - - - - - - - - - - - - - sextis

$\mathfrak{D} = \frac{2^6}{1 \ldots 9} \cdot \frac{3}{5} = \frac{1}{9450}$ - - - - - - - - - - - - octauis

$\mathfrak{E} = \frac{2^8}{1 \ldots 11} \cdot \frac{5}{3} = \frac{1}{93555}$ - - - - - - - - - - decimis

$\mathfrak{F} = \frac{2^{10}}{1 \ldots 13} \cdot \frac{691}{105}$ - - - - - - - - - - - duodecimis

$\mathfrak{G} = \frac{2^{12}}{1 \ldots 15} \cdot \frac{35}{1}$ - - - - - - - - - decimis quartis

$\mathfrak{H} = \frac{2^{14}}{1 \ldots 17} \cdot \frac{3617}{15}$ - - - - - - - - - decimis sextis

$\mathfrak{I} = \frac{2^{16}}{1 \ldots 19} \cdot \frac{43867}{21}$ - - - - - - - - - decimis octauis

$\mathfrak{K} = \frac{2^{18}}{1 \ldots 21} \cdot \frac{1222277}{55}$ - - - - - - - - - - vigesimis

$\mathfrak{L} = \frac{2^{20}}{1 \ldots 23} \cdot \frac{854513}{3}$ - - - - - - - - - viges. sec.

$\mathfrak{M} = \frac{2^{22}}{1 \ldots 25} \cdot \frac{1181820455}{273}$ - - - - - - - - - viges. quart.

$\mathfrak{N} =$

$\mathfrak{N} = \frac{2^{24}}{1 \cdot \cdot \cdot \cdot \cdot 27} \cdot \frac{76977927}{1}$ - - - - - - - - viges. sext.

$\mathfrak{O} = \frac{2^{26}}{1 \cdot \cdot \cdot \cdot \cdot 29} \cdot \frac{23749461029}{15}$ - - - - - - - viges. oct.

$\mathfrak{P} = \frac{2^{28}}{1 \cdot \cdot \cdot \cdot \cdot 31} \cdot \frac{8615841276005}{231}$ - - - - - - - - trigesimis

$\mathfrak{Q} = \frac{2^{30}}{1 \cdot \cdot \cdot \cdot \cdot 33} \cdot \frac{84802581453387}{85}$ - - - - - - - triges. sec.

$\mathfrak{R} = \frac{2^{32}}{1 \cdot \cdot \cdot \cdot \cdot 35} \cdot \frac{90219075042845}{3}$ - - - - - - triges. quart.

Praeparatio Formularum generalium ad alios vsus.

§. 20. Hactenus posuimus $m = 1$, nunc autem statuamus $m = \frac{n-1}{2}$, eritque

$$\frac{m\pi}{n} = \frac{(n-1)\pi}{2n} = \frac{1}{2}\pi - \frac{\pi}{2n}, \text{ vnde fit}$$

$$\sin.\frac{m\pi}{n} = \cos.\frac{\pi}{2n} \text{ et tang.} \frac{m\pi}{n} = \cot.\frac{\pi}{2n}.$$

Ipsae autem series ita se habebunt:

$$\frac{\pi}{2n\cos.\frac{\pi}{2n}} = \frac{1}{n-1} - \frac{1}{3n-1} + \frac{1}{5n-1} - \frac{1}{7n-1} + \frac{1}{9n-1} - \text{etc.}$$

$$+ \frac{1}{n+1} - \frac{1}{3n+1} + \frac{1}{5n+1} - \frac{1}{7n+1} + \frac{1}{9n+1} - \text{etc.}$$

$$\frac{\pi}{2n\cot.\frac{\pi}{2n}} = \frac{1}{n-1} + \frac{1}{3n-1} + \frac{1}{5n-1} + \frac{1}{7n-1} + \frac{1}{9n-1} + \text{etc.}$$

$$- \frac{n}{n+1} - \frac{1}{3n+1} - \frac{1}{5n+1} - \frac{1}{7n+1} - \frac{1}{9n+1} - \text{etc.}$$

Euolutio seriei prioris § 20.

§. 21. Contrahantur hic etiam bini termini analogi, ac prodibit haec series:

$$\frac{\pi}{2n\cos.\frac{\pi}{2n}} = \frac{2n}{nn-1} - \frac{6n}{9nn-1} + \frac{10n}{25nn-1} - \frac{14n}{49nn-1} + \text{etc.}$$

 siue

stae

$$\frac{\pi}{4n\cos.\frac{\pi}{2n}} = \frac{n}{nn-1} - \frac{3n}{9nn-1} + \frac{5n}{25nn-1} - \frac{7n}{49nn-1} + \text{etc.}$$

§. 22. Hic igitur omnes istae fractiones continentur in hac forma generali: $\frac{in}{i^2n^2-1}$, vbi i denotat omnes numeros impares. Haec autem fractio in seriem infinitam conversa praebet

$$\frac{1}{in} + \frac{1}{i^3n^3} + \frac{1}{i^5n^5} + \frac{1}{i^7n^7} + \frac{1}{i^9n^9} + \text{etc.}$$

Hinc igitur singulas fractiones per series euoluamus:

$$\frac{n}{nn-1} = \frac{1}{n} + \frac{1}{n^3} + \frac{1}{n^5} + \frac{1}{n^7} + \frac{1}{n^9} + \text{etc.}$$

$$-\frac{3n}{9nn-1} = -\frac{1}{3n} - \frac{1}{3^3n^3} - \frac{1}{3^5n^5} - \frac{1}{3^7n^7} - \frac{1}{3^9n^9} - \text{etc.}$$

$$\frac{5n}{25nn-1} = \frac{1}{5n} + \frac{1}{5^3n^3} + \frac{1}{5^5n^5} + \frac{1}{5^7n^7} + \frac{1}{5^9n^9} + \text{etc.}$$

$$-\frac{7n}{49nn-1} = -\frac{1}{7n} - \frac{1}{7^3n^3} - \frac{1}{7^5n^5} - \frac{1}{7^7n^7} - \frac{1}{7^9n^9} - \text{etc.}$$

etc. etc

quarum igitur serierum omnium summa est $\frac{\pi}{4n\cos.\frac{\pi}{2n}}$.

§. 23. Nunc etiam has series per columnas verticales colligamus, ac statuamus

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.} = \mathfrak{a}\frac{\pi}{2}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.} = \mathfrak{b}\frac{\pi^3}{2^3}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \frac{1}{11^5} + \text{etc.} = \mathfrak{c}\frac{\pi^5}{2^5}$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \frac{1}{11^7} + \text{etc.} = \mathfrak{d}\frac{\pi^7}{2^7}$$

etc. etc.

qui-

quibus positis aequatio nostra erit

$$\frac{\pi}{4n\cos.\frac{\pi}{2n}} = \frac{\mathfrak{a}\pi}{2n} + \frac{\mathfrak{b}\pi^3}{2^3n^3} + \frac{\mathfrak{c}\pi^5}{2^5n^5} + \frac{\mathfrak{d}\pi^7}{2^7n^7} + \text{etc.}$$

§. 24. Ponamus nunc $\frac{\pi}{2n} = x$, et aequatio nostra hanc induet formam:

$$\frac{x}{2\cos.x} = \mathfrak{a}x + \mathfrak{b}x^3 + \mathfrak{c}x^5 + \mathfrak{d}x^7 + \mathfrak{e}x^9 + \text{etc.}$$

Nunc igitur, si breuitatis gratia ponamus

$$\cos.x = \alpha - \beta xx + \gamma x^4 - \delta x^6 + \varepsilon x^8 - \text{etc.}$$

ita vt sit

$$\alpha = 1,\ \beta = \frac{1}{1.2},\ \gamma = \frac{1}{1\ldots4},\ \delta = \frac{1}{1\ldots6},\ \text{etc.}$$

si per hanc seriem vtrinque multiplicemus, orietur ista aequatio:

$$\left.\begin{array}{llllllll}
\frac{x}{2} = \alpha\mathfrak{a}x & +\alpha\mathfrak{b}x^3 & +\alpha\mathfrak{c}x^5 & +\alpha\mathfrak{d}x^7 & +\alpha\mathfrak{e}x^9 & +\alpha\mathfrak{f}x^{11} & +\alpha\mathfrak{g}x^{13} \\
 & -\beta\mathfrak{a} & -\beta\mathfrak{b} & -\beta\mathfrak{c} & -\beta\mathfrak{d} & -\beta\mathfrak{e} & -\beta\mathfrak{f} \\
 & & +\gamma\mathfrak{a} & +\gamma\mathfrak{b} & +\gamma\mathfrak{c} & +\gamma\mathfrak{d} & +\gamma\mathfrak{e} \\
 & & & -\delta\mathfrak{a} & -\delta\mathfrak{b} & -\delta\mathfrak{c} & -\delta\mathfrak{d} \\
 & & & & +\varepsilon\mathfrak{a} & +\varepsilon\mathfrak{b} & +\varepsilon\mathfrak{c} \\
 & & & & & -\zeta\mathfrak{a} & -\zeta\mathfrak{b} \\
 & & & & & & +\eta\mathfrak{a}
\end{array}\right\}\ \text{etc.}$$

§. 25. Singulis igitur potestatibus ad nihilum reductis nanciscemur sequentes determinationes:

$\mathfrak{a} = \frac{1}{2}$

$\mathfrak{b} = \beta\mathfrak{a}$

$\mathfrak{c} = \beta\mathfrak{b} - \gamma\mathfrak{a}$

 $\mathfrak{d} =$

$$\mathfrak{d} = \beta\mathfrak{c} - \gamma\mathfrak{b} + \delta\mathfrak{a}$$
$$\mathfrak{e} = \beta\mathfrak{d} - \gamma\mathfrak{c} + \delta\mathfrak{b} - \varepsilon\mathfrak{a}$$
$$\mathfrak{f} = \beta\mathfrak{e} - \gamma\mathfrak{d} + \delta\mathfrak{c} - \varepsilon\mathfrak{b} + \zeta\mathfrak{a}$$
etc. etc.

§. 26. Ope harum formularum iam olim summas istarum serierum exhibui, vnde valores pro praesentibus litteris $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, etc. ita reperientur determinati:

$\mathfrak{a} = \frac{1}{2}$ pro potestatibus I.

$\mathfrak{b} = \frac{1}{1.2} \cdot \frac{1}{2}$ - - - - - - - - - III.

$\mathfrak{c} = \frac{5}{1.2.3.4} \cdot \frac{1}{2}$ - - - - - - - - V.

$\mathfrak{d} = \frac{61}{1 \ldots\ldots 6} \cdot \frac{1}{2}$ - - - - - - - VII.

$\mathfrak{e} = \frac{1385}{1 \ldots\ldots 8} \cdot \frac{1}{2}$ - - - - - - - IX.

$\mathfrak{f} = \frac{50521}{1 \ldots\ldots 10} \cdot \frac{1}{2}$ - - - - - - - XI.

$\mathfrak{g} = \frac{2702765}{1 \ldots\ldots 12} \cdot \frac{1}{2}$ - - - - - - - XIII.

$\mathfrak{h} = \frac{199360981}{1 \ldots\ldots 14} \cdot \frac{1}{2}$ - - - - - - - XV.

$\mathfrak{i} = \frac{19391512145}{1 \ldots\ldots 16} \cdot \frac{1}{2}$ - - - - - - - XVII.

$\mathfrak{k} = \frac{2404879661671}{1 \ldots\ldots 18} \cdot \frac{1}{2}$ - - - - - - - XIX.

§. 27. Hinc igitur summas istarum serierum vsque ad potestatem vigesimam apponamus:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \frac{1}{1} \cdot \frac{\pi}{2^2}$$
$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \text{etc.} = \frac{1}{1.2} \cdot \frac{\pi^3}{2^4}$$
$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \text{etc.} = \frac{1}{1.2.3.4} \cdot \frac{\pi^5}{2^6}$$
$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \text{etc.} = \frac{61}{1 \ldots\ldots 6} \cdot \frac{\pi^7}{2^8}$$
$$1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \frac{1}{9^9} - \text{etc.} = \frac{1385}{1 \ldots\ldots 8} \cdot \frac{\pi^9}{2^{10}}$$

1 —

$$1-\frac{1}{3^{11}}+\frac{1}{5^{11}}-\frac{1}{7^{11}}+\frac{1}{9^{11}}-\text{ etc.}=\frac{50521}{1\ldots\ldots10}\cdot\frac{\pi^{11}}{2^{12}}$$

$$1-\frac{1}{3^{13}}+\frac{1}{5^{13}}-\frac{1}{7^{13}}+\frac{1}{9^{13}}-\text{ etc.}=\frac{2702765}{1\ldots\ldots12}\cdot\frac{\pi^{13}}{2^{14}}$$

$$1-\frac{1}{3^{15}}+\frac{1}{5^{15}}-\frac{1}{7^{15}}+\frac{1}{9^{15}}-\text{ etc.}=\frac{199360981}{1\ldots\ldots14}\cdot\frac{\pi^{15}}{2^{16}}$$

$$1-\frac{1}{3^{17}}+\frac{1}{5^{17}}-\frac{1}{7^{17}}+\frac{1}{9^{17}}-\text{ etc.}=\frac{19391512145}{1\ldots\ldots16}\cdot\frac{\pi^{17}}{2^{18}}$$

$$1-\frac{1}{3^{19}}+\frac{1}{5^{19}}-\frac{1}{7^{19}}+\frac{1}{9^{19}}-\text{ etc.}=\frac{2404879661671}{1\ldots\ldots18}\cdot\frac{\pi^{19}}{2^{20}}$$

etc. etc.

Euolutio feriei pofterioris § 20.

§. 28. Binis terminis analogis contractis haec feries hanc induet formam:

$$\frac{\pi}{4n\cot.\frac{\pi}{2n}}=\frac{1}{nn-1}+\frac{1}{9nn-1}+\frac{1}{25nn-1}+\frac{1}{49nn-1}+\text{ etc.}$$

quae fractiones in feries euolutae dabunt:

$$\frac{1}{nn-1}=\frac{1}{nn}+\frac{1}{n^4}+\frac{1}{n^6}+\frac{1}{n^8}+\frac{1}{n^{10}}+\text{ etc.}$$

$$\frac{1}{9nn-1}=\frac{1}{9nn}+\frac{1}{9^2n^4}+\frac{1}{9^3n^6}+\frac{1}{9^4n^8}+\frac{1}{9^5n^{10}}+\text{ etc.}$$

$$\frac{1}{25nn-1}=\frac{1}{25nn}+\frac{1}{25^2n^4}+\frac{1}{25^3n^6}+\frac{1}{25^4n^8}+\frac{1}{25^5n^{10}}+\text{ etc.}$$

etc. etc.

quarum igitur omnium fumma eft $\frac{\pi}{4n}\text{tang.}\frac{\pi}{2n}$.

§. 29. Quo nunc has feries verticaliter colligere queamus, ftatuamus:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\text{ etc.}=\mathfrak{A}'\frac{\pi\pi}{2^2}$$

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{ etc.}=\mathfrak{B}'\frac{\pi^4}{2^4}$$

$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{ etc.}=\mathfrak{C}'\frac{\pi^6}{2^6}$$

etc. etc.

Vnde

Vnde aequatio nostra fiet

$$\frac{\pi}{4n}\,\text{tang.}\,\frac{\pi}{2n} = \frac{\mathfrak{A}'\pi^2}{2^2\,n^2} + \frac{\mathfrak{B}'\pi^4}{2^4\,n^4} + \frac{\mathfrak{C}'\pi^6}{2^6\,n^6} + \frac{\mathfrak{D}'\pi^8}{2^8\,n^8} + \text{ etc.}$$

§. 30. Ponamus nunc $\frac{\pi}{2n} = x$, vt aequatio nostra fiat

$$\frac{x}{2}\,\text{tang.}\,x = \mathfrak{A}'xx + \mathfrak{B}'x^4 + \mathfrak{C}'x^6 + \text{ etc.}$$

vnde erit

$$\text{tang.}\,x = 2\,\mathfrak{A}'x + 2\,\mathfrak{B}'x^3 + 2\,\mathfrak{C}'x^5 + 2\,\mathfrak{D}'x^7 + \text{ etc.}$$

cuius seriei loco scribamus litteram t, vt sit tang. $x = t$, hincque differentiando $dx = \frac{dt}{1+tt}$, ideoque habebimus $\frac{dt}{dx} = 1 + tt$. Est vero

$$\frac{dt}{dx} = 2\,\mathfrak{A}' + 2.\,3\,\mathfrak{B}'xx + 2.\,5\,\mathfrak{C}'x^4 + 2.\,7\,\mathfrak{D}'x^6 + \text{ etc.}$$

§. 31. Eodem modo facta euolutione erit

$$1 + tt = 1 + 4\,\mathfrak{A}'\mathfrak{A}'xx + 8\,\mathfrak{A}'\mathfrak{B}'x^4 + 8\,\mathfrak{A}'\mathfrak{C}'x^6 + 8\,\mathfrak{A}'\mathfrak{D}'x^8 + \text{ etc.}$$
$$+ 4\,\mathfrak{B}'\mathfrak{B}' \quad + 8\,\mathfrak{B}'\mathfrak{C}' + \text{ etc.}$$

Hinc igitur sequentes deducuntur determinationes:

$$\mathfrak{A}' = \tfrac{1}{2}$$
$$\mathfrak{B}' = \tfrac{2}{3}.\,\mathfrak{A}'\mathfrak{A}'$$
$$\mathfrak{C}' = \tfrac{2}{5}.\,2\,\mathfrak{A}'\mathfrak{B}'$$
$$\mathfrak{D}' = \tfrac{2}{7}(2\,\mathfrak{A}'\mathfrak{C}' + \mathfrak{B}'\mathfrak{B}')$$
$$\mathfrak{E}' = \tfrac{2}{9}(2\,\mathfrak{A}'\mathfrak{D}' + 2\,\mathfrak{B}'\mathfrak{C}')$$
$$\mathfrak{F}' = \tfrac{2}{11}(2\,\mathfrak{A}'\mathfrak{E}' + 2\,\mathfrak{B}'\mathfrak{D}' + \mathfrak{C}'\mathfrak{C}')$$
$$\mathfrak{G}' = \tfrac{2}{13}(2\,\mathfrak{A}'\mathfrak{F}' + 2\,\mathfrak{B}'\mathfrak{E}' + 2\,\mathfrak{C}'\mathfrak{D}')$$

etc. etc.

§. 32. Istae determinationes fere prorsus conueniunt cum iis, quas supra pro litteris $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. inuenimus; totum enim discrimen reperitur in coëfficientibus numericis. Neutiquam vero opus est istos valores seorsim com-

computare, cum ii iam ex superioribus facillime deduci queant. Cum enim sit $\frac{\mathfrak{A}'}{2^2} = \frac{\mathfrak{A}(2^2-1)}{2^2}$, erit $\mathfrak{A}' = (2^2-1)\mathfrak{A}$. Simili modo erit

$$\mathfrak{B}' = (2^4-1)\mathfrak{B},\quad \mathfrak{C}' = (2^6-1)\mathfrak{C},\quad \mathfrak{D}' = (2^8-1)\mathfrak{D},\ \text{etc.}$$

Conclusio.

§. 33. In gratiam eorum, qui hos valores penitus numerice per fractiones decimales exprimere voluerint, subiungamus sequentem tabulam, in qua omnes potestates ipsius π per fractiones decimales sunt euolutae, vbi loco $\frac{\pi}{2}$ scripsimus q.

$$\begin{aligned}
q &= 1,57079,63267,94896,61923,13216,916\\
\frac{q^2}{1.2} &= 1,23370,05501,36169,82735,43113,745\\
\frac{q^3}{1.2.3} &= 0,64596,40975,06246,25365,57565,636\\
\frac{q^4}{1\ldots4} &= 0,25366,95079,01048,01363,65633,659\\
\frac{q^5}{1\ldots5} &= 0,07969,26262,46167,04512,05055,487\\
\frac{q^6}{1\ldots6} &= 0,02086,34807,63352,96087,30516,364\\
\frac{q^7}{1\ldots7} &= 0,00468,17541,35318,68810,06854,633\\
\frac{q^8}{1\ldots8} &= 0,00091,92602,74839,42658,02417,158\\
\frac{b^9}{1\ldots9} &= 0,00016,04411,84787,35982,18726,605\\
\frac{q^{10}}{1\ldots10} &= 0,00002,52020,42373,06060,54810,526\\
\frac{q^{11}}{1\ldots11} &= 0,00000,35988,43235,21208,53404,580\\
\frac{q^{12}}{1\ldots12} &= 0,00000,04710,87477,88181,71503,665\\
\frac{q^{13}}{1\ldots13} &= 0,00000,00569,21729,21967,92681,170\\
\frac{q^{14}}{1\ldots14} &= 0,00000,00063,86603,08379,18522,408
\end{aligned}$$

$$\frac{q^{15}}{1 \ldots \ldots 15} = 0,\ 00000,\ 00006,\ 68803,\ 51098,\ 11467,\ 225$$

$$\frac{q^{16}}{1 \ldots \ldots 16} = 0,\ 00000,\ 00000,\ 65659,\ 63114,\ 97947,\ 230$$

$$\frac{q^{17}}{1 \ldots \ldots 17} = 0,\ 00000,\ 00000,\ 06066,\ 93573,\ 11061,\ 950$$

$$\frac{q^{18}}{1 \ldots \ldots 18} = 0,\ 00000,\ 00000,\ 00529,\ 44002,\ 00734,\ 620$$

$$\frac{q^{19}}{1 \ldots \ldots 19} = 0,\ 00000,\ 00000,\ 00043,\ 77065,\ 46731,\ 370$$

$$\frac{q^{20}}{1 \ldots \ldots 20} = 0,\ 00000,\ 00000,\ 00003,\ 43773,\ 91790,\ 981$$

$$\frac{q^{21}}{1 \ldots \ldots 21} = 0,\ 00000,\ 00000,\ 00000,\ 25714,\ 22892,\ 855$$

$$\frac{q^{22}}{1 \ldots \ldots 22} = 0,\ 00000,\ 00000,\ 00000,\ 01835,\ 99165,\ 212$$

$$\frac{q^{23}}{1 \ldots \ldots 23} = 0,\ 00000,\ 00000,\ 00000,\ 00125,\ 38995,\ 403$$

$$\frac{q^{24}}{1 \ldots \ldots 24} = 0,\ 00000,\ 00000,\ 00000,\ 00008,\ 20675,\ 327$$

$$\frac{q^{25}}{1 \ldots \ldots 25} = 0,\ 00000,\ 00000,\ 00000,\ 00000,\ 51564,\ 550$$

$$\frac{q^{26}}{1 \ldots \ldots 26} = 0,\ 00000,\ 00000,\ 00000,\ 00000,\ 03115,\ 285$$

$$\frac{q^{27}}{1 \ldots \ldots 27} = 0,\ 00000,\ 00000,\ 00000,\ 00000,\ 00181,\ 239$$

$$\frac{q^{28}}{1 \ldots \ldots 28} = 0,\ 00000,\ 00000,\ 00000,\ 00000,\ 00010,\ 165$$

$$\frac{q^{29}}{1 \ldots \ldots 29} = 0,\ 00000,\ 00000,\ 00000,\ 00000,\ 00000,\ 549$$

$$\frac{q^{30}}{1 \ldots \ldots 30} = 0,\ 00000,\ 00000,\ 00000,\ 00000,\ 00000,\ 026$$

$$\frac{q^{31}}{1 \ldots \ldots 31} = 0,\ 00000,\ 00000,\ 00000,\ 00000,\ 00000,\ 000$$

Hae quidem poteſtates diuiſae ſunt per certos numeros, qui autem plerumque ſunt ii ipſi, per quos eaedem poteſtates ipſius π in ſuperioribus formulis diuiſi occurrunt, vnde euolutio in fractiones decimales eo facilior redditur.

DE

DE INSIGNI PROMOTIONE SCIENTIAE NVMERORVM.

§. 1.

Eximia omnino ſunt, quae celeberr. *La Grange* in Comment. Academiae Regiae Boruſſicae pro Anno 1773 de diuiſoribus formulae generaliſſimae $Btt + Ctu + Duu$ demonſtrauit, et maximam lucem in ſcientia numerorum, quae etiamnunc tantis tenebris eſt inuoluta, accendunt. Ob hoc ipſum autem, quod iſta tractatio maxime eſt generalis, ii qui non ſatis ſunt exercitati in huiusmodi ſpeculationibus, non parum difficultatis offendunt, neque vim talium ſublimium demonſtrationum ſatis perſpicere valent. Quamobrem haud inutile erit omnia momenta, quibus hae demonſtrationes innituntur, diligentius explicare atque ad formulas magis ſpeciales accommodare, quandoquidem hoc modo omnia facilius intelligi poterunt. Deinde imprimis accuratius exponam, quantum firmamentum hinc plurimis theorematibus, quorum veritatem per ſolam inductionem mihi quidem cognoſcere licuit, afferri poſſit, vnde multo clarius patebit, quantum adhuc ad eorum perfectam demonſtrationem deſideretur.

Lemma.

§. 2. Si p et q fuerint numeri inter ſe primi, tum omnes plane numeri in hac forma generali $\alpha p \pm \beta q$ comprehendi poſſunt, idque infinitis modis. Huius lemmatis demonſtratio per ſe facilis paſſim inuenitur.

Problema I.

§. 3. *Si* p *et* q *ſint numeri inter ſe primi*, n *vero denotet numerum quemcunque datum, ſiue poſitiuum ſiue negatiuum, inuenire omnes diuiſores huius formulae:* $pp + nqq$.

Solutio.

Denotet D diuiſorem quemcunque numeri in hac forma $pp + nqq$ contenti, ſitque d quotus ex hac diuiſione ortus, ita vt ſit $Dd = pp + nqq$. Hic iam ſtatim euidens eſt, numerum d ad q fore primum; ſi enim q haberet diuiſorem communem, eundem quoque p habere deberet, contra hypotheſin, quamobrem numerus p per d et q ita exprimi poterit, vt ſit $p = \alpha d \pm \beta q$, quo valore ſubſtituto fiet

$$Dd = \alpha\alpha dd \pm 2\alpha\beta dq + (\beta\beta + n)qq$$

ideoque diuiſor

$$D = \alpha\alpha d \pm 2\alpha\beta q + \left(\frac{\beta\beta + n}{d}\right)qq,$$

vbi ergo $\frac{\beta\beta + n}{d}$ erit numerus integer, qui ſit $= h$, ita vt habeatur

$$D = \alpha\alpha d \pm 2\alpha\beta q + hqq,$$

pro qua forma ſcribamus

$$D = frr \pm gqr + hqq,$$

ita

ita vt sit $f = d$, $r = \alpha$, $g = 2\beta$ et ob $h = \frac{\beta\beta + n}{d}$ erit $fh = \beta\beta + n$; hineque fiet $4fh - gg = 4n$. Hinc igitur patet, omnes diuisores formae $pp + nqq$ semper contineri in hac forma:

$$D = frr \pm gqr + hqq,$$

dummodo fuerit $4fh - gg = 4n$. Ac viciſſim, ſi fuerit

$$D = frr \pm gqr + hqq, \text{ erit}$$

$$4df = 4ffrr \pm 4fgqr + 4fhqq;$$

ideoque ob $4fh = 4n + gg$, erit

$$4Df = (2fr \pm gq)^2 + 4nqq,$$

quae forma a propoſita non diſcrepat, ſi modo diuidatur per 4.

Corollarium 1.

§. 4. In genere igitur omnes diuiſores formulae propoſitae $pp + nqq$ comprehendere licebit in iſta formula latiſſime patente: $frr + grs + hss$, dummodo fuerit $4fh - gg = 4n$, ſiue $fh - \frac{1}{4}gg = n$; vnde patet, innumeras huiusmodi formulas exhiberi poſſe, quoniam numerum g pro lubitu accipere licet. Ex eo autem numeros f et h ita definiri oportet, vt fiat $4fh = 4n + gg$.

Scholion.

§. 5. Quoniam autem innumerabiles huiusmodi formulas: $frr + grs + hss$, exhibere licet, in quibus ſit $4fh - gg = n$, parum hinc lucri ad noſtrum inſtitutum afferri videtur. Propoſito enim quocunque numero D, ad diiudicandum, vtrum eſſe poſſit diuiſor formae $pp + nqq$, omnes illae innumerabiles formulae conſiderari deberent,

 num

num forte iste numerus D in quapiam illarum contineatur. Praecipuum igitur inuentum, quod Illustri *la Grange* acceptum referre debemus, in hoc consistit, quod infinitam illam huiusmodi formularum multitudinem ad exiguum numerum pro quouis casu reuocare docuit, id quod in sequente problemate exponamus.

Problema II.

§. 6. Formam generalem diuisorum ante inuentam, f r r + g r s + h s s, *in qua sit* 4 f h − g g = n, *in aliam eiusdem formae,* f′ t t + g′ t u + h′ u u, *transmutare, in qua sit* g′ < f′ *vel* h′; *manente proprietate* 4 f′ h′ − g′ g′ = n.

Solutio.

Ponamus esse $f < h$ et numerum g quantumuis esse maiorem quam f, ac statuamus $r = t - \alpha s$, quo valore substituto orietur ista forma:

$$ftt + (g - 2\alpha f)ts + (\alpha\alpha f - \alpha g + h)ss;$$

vbi manifesto α ita assumi poterit, vt fiat $g - 2\alpha f < f$, vbi quidem animaduertendum est, nihil referre, vtrum $g - 2\alpha f$ prodeat positiuum an negatiuum. Statuatur igitur $g - 2\alpha f = \pm g'$, ita vt certe sit $g' < f'$, tum vero ob analogiam loco f scribatur f' et $\alpha\alpha f - \alpha g + h = h'$ eritque

$$4f'h' - g'g' = 4fh - gg = n.$$

Hoc igitur modo forma proposita reducta est ad hanc:

$$f'tt \pm g'ts + h'ss,$$

in

in qua certe eſt $g' < f'$. Quod ſi iam eueniat vt g' adhuc maius fuerit quam h', tum ſimili modo iſta formula in aliam transformari poterit, in qua coëfficiens medius vtrovis extremo ſit minor, vnde patet formam propoſitam

$$frr \pm grs + hss$$

ſemper in aliam ſimilis formae

$$f'tt \pm g'tu + h'uu$$

conuerti poſſe, in qua g' minus ſit quam f' et h', ſimulque etiam fiat $4f'h' - g'g' = n$.

Corollarium 1.

§. 7. Hoc igitur modo infinita multitudo formularum $frr + grs + hss$, in qua $4fh - gg = n$ plerumque ad ſatis exiguum numerum reduci poteſt, dum ſcilicet omnes illae formulae excludi poſſunt, in quibus coëfficiens medius g maior eſt alterutro extremorum.

Corollarium 2.

§. 8. Cum igitur ſit tam $f > g$ quam $h > g$, erit $4fh > 4gg$. Sit igitur $4fh = 4gg + \Delta$, et cum eſſe debeat $4fh - gg = 4n$, erit $3gg + \Delta = 4n$, ideoque $3gg < 4n$, hinc ergo $g < \sqrt{\frac{4n}{3}}$; quamobrem loco g ſucceſſiue eos tantum valores aſſumſiſſe ſufficiet, qui ſunt minores quam $\sqrt{\frac{4n}{3}}$, ex quibus ſingulis facile colligentur valores litterarum f et h ex aequatione $4fh = gg + n$, quo facto omnes plane diuiſores formae $pp + nqq$ certe continebuntur in quapiam harum formularum ſimpliciorum.

Scho-

Scholion.

§. 9. Quoniam aequatio $4fh - gg = 4n$ locum habere nequit, nisi g sit numerus par, pro g statim scribamus $2g$, vt forma d sit $frr + 2grs + hss$, existente $fh - gg = n$, quae ergo forma semper ita reduci potest vt sit $2g < f$ vel $< h$. Haec autem reductio commodissime per gradus institui potest, dum loco α in superiori reductione scribitur vnitas. Ita si fuerit diuisor

$$D = frr + 2grs + hss,$$

erit quoque

$$D = f'rr + 2g'rs + h'ss,$$

existente duplici modo vel

$$f' = f, \; g' = f - g \text{ et } h' = f - 2g + h,$$

vel etiam

$$h' = h, \; g' = h - g \text{ et } f' = f - 2g + h,$$

quoniam membra extrema inter se commutare licet. Quod si hic nondum fuerit $2g' < f'$ seu $2g' < h'$, ista operatio tam diu continuari debet, donec fiat $2g < f$ vel h; vbi notandum, in his formulis terminum medium $2grs$ tam positiuum quam negatiuum accipi posse, propterea quod numeri r et s denotare possunt omnes numeros integros siue positiuos siue negatiuos. His igitur praemissis inuestigemus omnes diuisores primos numerorum vel in hac forma: $pp + nqq$, vel in hac: $pp - nqq$ contentorum; si quidem diuisores compositi ex primis componuntur, ita vt cognitis omnibus diuisoribus primis simul omnes compositi habeantur.

Pro-

Problema III.

§. 10. *Inuenire omnes diuisores primos numerorum in hac forma:* $pp+nqq$, *contentorum*, *existentibus numeris* p *et* q *tam inter se primis quam respectu numeri* n.

Solutio.

I. Quia enim hic de diuisoribus primis tantum sermo est, nisi p esset quoque primus ad n, formula $pp+nqq$ etiam admitteret omnes diuisores numeri n, qui propterea nullam inuestigationem requirunt et sponte se produnt. Sit ergo D diuisor quicunque formae $pp+nqq$, ac modo vidimus semper fore

$$D = frr + 2grs + hss,$$

existente $fh - gg = n$, ita vt sit tam $2g < f$ quam $2g < h$; quare cum hinc sit $f > 2g$ et $h > 2g$, erit $fh > 4gg$ Sit igitur $fh = 4gg + \Delta$, et quia $fh - gg = n$, erit $3gg + \Delta = n$ ideoque $gg < \frac{n}{3}$ et $g < \sqrt{\frac{n}{3}}$. Hac igitur conditione multitudo formarum pro diuisore D ad eo minorem numerum reducitur, quo minor fuerit numerus n. Cum igitur sit

$$D = frr + 2grs + hss, \text{ erit}$$

$$Df = ffrr + 2fgrs + fhss,$$

ideoque ob $fh = gg + n$ fiet

$$Df = (fr + gs)^2 + nss,$$

quae est ipsa forma proposita. Simili modo permutatis litteris f et h erit quoque

$$Dh = (hs + gr)^2 + nrr,$$

vnde patet, si fuerit Df numerus formae $pp+nqq$, tum etiam productum Dh fore eiusdem formae, ita vt suf-

 ficiat

ficiat alterutram inueniſſe. Pro quouis ergo caſu quaerantur omnes valores litterae f, qui ſint f, f', f'', f''', etc. atque omnes diuiſores primi D ita erunt comparati, vt vel D, vel Df, vel Df', vel Df'', etc. ſint numeri formae $pp + nqq$. Haecque ſequuntur ex demonſtrationibus Illuſtris *la Grange*.

II. Haec igitur coniungamus cum iis, quae iam olim de formis horum diuiſorum primorum ſum commentatus, vbi oſtendi, omnes hos diuiſores comprehendi poſſe in huiusmodi expreſſione: $4ni + a$, dum ſcilicet a denotat certos numeros primos ad $4n$ ſimulque minores quam $4n$, vbi tantum ſemiſſis talium numerorum occurrit, reliquis hinc prorſus excluſis. Vnde ſi α denotet hos numeros excluſos, affirmari poterit, nullos numeros, in forma $4ni + \alpha$ contentos, eſſe poſſe diuiſores formae $pp + nqq$. Iſtae autem formae egregie conueniunt cum praecedentibus. Si enim fuerit $D = pp + nqq$, alteruter numerorum p et q debet eſſe impar, ideoque q vel par vel impar. Sit primo q par, ideoque qq numerus formae $4i$, fiet $D = 4ni + pp$. Vnde patet, litteram illam a complecti omnes numeros quadratos impares et primos ad $4n$, ſiue reſidua, quae ex diuiſione horum quadratorum, per $4n$ facta, remanent. Sin autem fuerit q numerus impar, ideoque qq formae $4i + 1$, hinc fiet $D = 4ni + pp + n$. Vnde patet, litteram a etiam complecti omnes numeros formae $pp + n$, qui quidem ad $4n$ ſint primi, vel eorum reſidua ex diuiſione per $4n$ remanentia. Iidem vero etiam numeri pro a reſultant, ſi fuerit Df numerus formae $pp + nqq$, id quod in exemplis facilius oſtendi poterit.

III. His expositis, cum forma $4ni + a$ complectatur omnes diuisores formae $pp + nqq$; altera autem forma $4ni + \alpha$ nullos diuisores in se inuoluat, ex priori forma $4ni + a$ excludi debebunt omnes numeri diuisibiles per quempiam numerum formae $4ni + \alpha$. Quod si igitur demonstrari posset, hoc modo ex formula $4ni + a$ omnes plane numeros excludi, qui nequeunt esse diuisores formae $pp + nqq$, tum manifesto sequeretur, omnes numeros primos formae $4ni + a$ certe esse diuisores formae $pp + nqq$, quandoquidem tantum numeros compositos hoc modo exclusimus. Totum ergo negotium huc redit, vt demonstretur, formulam $4ni + \alpha$ omnes plane continere numeros primos, qui nequeunt esse diuisores formae $pp + nqq$; quod si demonstrari posset, nihil amplius in hoc genere desideraretur.

Corollarium 1.

§. 11. Pro quouis ergo numero n omnes numeri ipso $4n$ minores ad eumque primi in duas classes distribuentur, quarum alteram littera a, alteram vero littera α designauimus, ita vt formula $4ni+a$ contineat omnes diuisores formae $pp + nqq$, altera vero formula, $4ni + \alpha$, diuisores illos penitus excludat, neque vllus numerus istius formae vnquam esse possit diuisor formulae $pp + nqq$. Multitudo autem numerorum vtriusque classis semper est eadem; scilicet si multitudo omnium numerorum minorum quam numerus $4n$ ad eumque primorum fuerit $= 2\lambda$ (semper enim iste numerus est par). Prior forma a continet λ numeros, totidemque etiam continebit altera forma α.

Corollarium 2.

§. 12. Circa has formulas: $4ni + a$ et $4ni + \alpha$, iam olim demonstraui, si numeri a et a' in priore classe occurrant, tum ibi quoque occurrere productum aa', id quod etiam de pluribus numeris huius classis est intelligendum, qui si fuerint a, a', a'', a''' etc. etiam producta tam ex binis quam pluribus horum numerorum, atque adeo etiam omnes eorum potestates in eadem classe reperientur, postquam scilicet per $4n$ diuisi ad residua minora quam $4n$ fuerint reducti. Deinde etiam demonstraui, si α fuerit numerus posterioris classis, tum in eadem quoque reperiri debere numeros $a\alpha$, $a'\alpha$, $a''\alpha$, $a'''\alpha$, etc. Vnde patet, multitudinem numerorum posterioris classis minorem esse non posse quam primae classis. Quod autem multitudo vtrinque sit prorsus aequalis. id etiam facile demonstrari potest. Tum vero etiam hoc certum est, si α, α', α'', α''', etc. fuerint numeri posterioris classis, tum tam eorum quadrata quam eorum producta ex binis in priorem classem ingredi, producta autem ex ternis iterum in classe posteriore reperiri.

Corollarium 3.

§. 13. Omnia igitur, quae adhuc in hoc genere desiderari possunt, huc redeunt, vt demonstretur, classem $4ni + \alpha$ omnes continere numeros primos, qui nequeant esse diuisores formae $pp + nqq$; tum enim euictum erit, omnes numeros primos formae prioris $4ni + a$ certe esse diuisores cuiuspiam numeri formae $pp + nqq$.

Problema IV.

§. 14. *Inuenire omnes diuisores primos numerorum in hac forma:* $pp - nqq$ *contentorum, vbi quidem, vt*

ante

ante, p *et* q *non solum sint primi inter se, sed etiam primi ad* n.

Solutio.

I. Conditio, quod p sit etiam primus ad n, ideo tantum hic adiicitur, quia alias etiam omnes diuisores numeri n hic in censum venirent, quos tamen hic excludimus, vtpote per se manifestos. Hic igitur primo patet, si fuerit D diuisor primus formae $pp - nqq$, tum etiam fore diuisorem formae $nqq - pp$, siquidem fuerit nqq maius quam pp. Nam si fuerit D diuisor formae $pp - nqq$, erit quoque diuisor formae $npp - nnqq$, quae forma, si loco nq scribamus r, abit in hanc: $npp - rr$. Deinde eodem modo vt ante patet, semper fore

$$D = frr + 2grs + hss,$$

existente $fh - gg = -n$, hancque formam semper ita reduci posse, vt fiat $2g < f$ simulque $2g < h$, vbi quidem signa numerorum f et h non respiciuntur, si forte alterutrum membrum fiat negatiuum; quare cum, ob $f > 2g$ et $h > 2g$, sit $fh > 4gg$, euidens est fieri non posse

$$fh - gg = -n$$

nisi vel f vel h fuerit negatiuum, vnde forma diuisoris ita debet constitui, vt sit

$$D = frr + 2grs - hss$$

fierique debet $-fh - gg = -n$, siue $fh + gg = +n$. Quoniam igitur $fh > 4gg$, necesse est vt sit $5gg < n$, ideoque $g < \sqrt{\tfrac{n}{5}}$, ita vt hoc casu pauciores valores pro g relinquantur. Tum autem erit

$$Df = ffrr + 2grs - fhss, \text{ siue}$$

$$Df = (fr + gs)^2 - nss$$

 quae

quae eſt forma ipſa propoſita. Porro autem erit

$$Dh = nrr - (gr - hs)^2,$$

quae eſt forma noſtra inuerſa $npp - qq$. Hinc igitur intelligitur, ſi fuerit Df numerus formae $pp - nqq$, tum eo ipſo formulam Dh fore numerum formae $npp - qq$.

II. Accommodemus haec etiam ad eam formam diuiſorum, quam olim exhibui; ac primo quidem ſi fuerit $D = pp\ nqq$, pro caſibus quibus q eſt numerus par, ideoque qq formae $4i$, fiet $D = pp - 4ni$; vnde ſi ponatur $D = 4ni + a$, ob $pp > 4ni$, ſi ponatur $pp = 4nk + b$ prodibit talis forma: $D = 4ni + b$, ita vt ſit $a = b$, ideoque omnes numeros quadratos ad $4n$ primos in ſe complectatur. Sin autem ſit q numerus impar, ideoque qq formae $4i + 1$, fiet $D = pp - n - 4ni$, poſitoque iterum $pp = 4nk + b$, prodit $D = 4ni + b - n$, ita vt hoc caſu ſit $a = b - n$, vbi b denotare poteſt omnes numeros quadratos, vel reſidua inde orta. Simili modo ſi fuerit $D = npp - qq$, euidens eſt, valores pro a hinc prodituros praecedentium fore negatiuos, ita vt a comprehendat omnes numeros quadratos, deinde etiam omnes numeros formae $pp - n$, tam poſitiue quam negatiue ſumtos; quamobrem forma omnium diuiſorum ita exhiberi poterit, vt ſit $4ni \pm a$, forma autem pro numeris ex claſſe diuiſorum exeluſis erit $4ni \pm \alpha$, quorum multitudo aequalis eſt priori, ſcilicet α ſemper totidem ſortietur valores, quot habet littera a.

III. Quo igitur etiam in hoc genere nihil amplius deſiderari queat, id tantum ſupereſt, vt demonſtretur, formam

mam posteriorem $4ni \pm \alpha$ omnes plane continere numeros primos, qui nunquam esse queant diuisores vllius numeri vel formae $pp - nqq$, vel $npp - qq$.

Corollarium 1.

§. 15. De his binis formulis: $4ni \pm a$ et $4ni \pm \alpha$, quarum illa omnes diuisores inuoluit, haec vero excludit, eadem valent, quae ante sunt tradita. Scilicet si a, a', a'', etc. ad priorem classem pertineant, ibidem quoque reperientur tam omnes potestates quam producta ex binis pluribusue horum numerorum; tum vero si α sit numerus posterioris classis, ibidem quoque occurrent omnes numeri $a\alpha$, $a'\alpha$, $a''\alpha$, etc., ita vt multitudo horum numerorum minor esse nequeat quam prioris classis.

Corollarium 2.

§. 16. Quoniam littera a complectitur omnia quadrata, ante omnia eius valor erit $= 1$, tum vero etiam 9, 25, etc. nisi numerus n habeat diuisorem vel 3, vel 5, etc. His enim casibus ista quadrata excludi oportet, quia alioquin forma $4ni \pm a$ numerus primus fieri non posset.

Scholion.

§. 17. His igitur generalibus praeceptis expositis omnia clariora euadent, si casus particulares euoluamus; hic enim plura adhuc occurrent, quae in genere attingere non licuit. Sufficiet autem id in aliquibus exemplis ostendisse, quibus pertractatis non difficile erit tabulam construere, quae pro omnibus casibus formas diuisorum primorum exhibeat.

Exem-

Exemplum 1.

§. 18. *Inuenire omnes diuisores primos numerorum in formula* $pp + nqq$ *contentorum, dum scilicet pro* p *et* q *assumatur numeri inter se primi.*

Solutio.

Posito diuisore primo

$$D = frr + 2grs + hss,$$

ob $n = 1$ debet esse $fh = gg + 1$, tum vero $g < \sqrt{\tfrac{1}{3}}$; vnde patet, pro g alium valorem assumi non posse praeter 0; tum autem erit $fh = 1$ ideoque tam $f = 1$ quam $h = 1$, sicque omnes diuisores in hac forma $D = rr + ss$ continebuntur, ita vt summa duorum quadratorum alios diuisores non admittat, nisi qui ipsi sint summae duorum quadratorum. Altera autem forma diuisorum erit $4i + 1$, et excludentur omnes numeri formae $4n + 3$ siue $4n - 1$. Quod si ergo demonstrari posset, formulam $4i - 1$ omnes plane continere numeros primos, qui nequeunt esse diuisores formae $pp + qq$, tum simul demonstratum esset, etiam omnes diuisores primos formae $4i + 1$ fore summam duorum quadratorum. Hoc autem iam dudum a me post Fermatium est demonstratum.

Exemplum 2.

§. 18. *Inuenire omnes diuisores primos formae* $pp - qq$.

Solutio.

Hoc exemplum ad problema quartum refertur, estque $n = 1$, et quia debet esse $g < \sqrt{\tfrac{1}{3}}$, necessario fieri oportet

$g =$

$g = 0$, ideoque $fh = 1$, vnde oritur haec forma diuisorum: $D = rr - ss$, quae vtique continet omnes plane numeros primos excepto binario. Quamquam enim haec forma habet factores $r + s$ et $r - s$, tamen continet omnes primos, si fuerit $r - s = 1$, cuius ratio est peculiaris. Id etiam altera diuisorum forma declarat, qua, ob $a = 1$, fit $4i \pm 1$, in qua omnes plane numeri impares continentur, ita vt hoc casu nulli excludantur, alteraque forma $4i + \alpha$ hoc solo casu nullum locum habeat. Ceterum hic casus proprie huc non pertinet, quia diuisores formae $pp - qq$ per se constant.

Exemplum 3.

§. 19. *Inuenire omnes diuisores primos formae* $pp + 2qq$.

Solutio.

Hic casus pertinet ad problema tertium, existente $n = 2$, vnde cum debeat esse $g < \sqrt{\frac{2}{3}}$, erit $g = 0$, ideoque $fh = 2$, hinc forma diuisorum erit $\pm rr + 2ss$. Vnde patet, numeros formae $pp + 2qq$ alios non admittere diuisores, nisi qui sint eiusdem formae, quod quidem etiam iam dudum est demonstratum. Altera autem forma $D = 8i + a$, ob $a = pp$, vel etiam $a = pp + 2$, pro a hos dat valores: 1 et 3, ita vt omnes diuisores formae $pp + 2qq$ sint vel $8i + 1$ vel $8i + 3$. Formae ergo, quae ex classe diuisorum excluduntur, sunt $8i + 5$ et $8i + 7$, quas igitur sub forma $8i + \alpha$ complecti oportet. Quod si ergo demonstrari posset, solos numeros primos harum formarum ex classe diuisorum excludi, simul demonstratum esset, omnes numeros primos priorum formarum $8n + 1$ et $8n + 3$ contineri in formula $pp + 2qq$, id quod quidem iam est ostensum.

sum. Ceterum binae posteriores formulae etiam ita exprimi possunt: $8i-1$ et $8i-3$, ita vt valores ipsius α sint negatiui ipsius a, id quod in genere de diuisoribus formae $pp+nqq$ est tenendum.

Exemplum 4.

§. 20. *Inuenire omnes diuisores primos formae* $pp-2qq$ *siue* $2pp-qq$.

Solutio.

Ex problemate quarto est $n=2$, ideoque, ob $g<\sqrt{\frac{2}{3}}$, erit iterum $g=0$ et $fh=2$, vnde pro diuisoribus erit $D=rr-2ss$, vel etiam $D=2rr-ss$; vnde patet has formas nullos alios diuisores admittere, nisi qui ipsi sint eiusdem formae. Pro forma autem $D=8i+a$, quia est $a=pp$, vel etiam $a=pp-2$, valores pro a erunt ± 1, ergo omnes dinisores continebuntur in forma $8i\pm 1$; excluduntur ergo omnes numeri formae $8i\pm 3$. Vnde si soli numeri primi formae $8i\pm 3$ ex classe diuisorum excludantur, necesse est, vt omnes numeri primi formae $8i\pm 1$ in forma proposita contineantur.

Corollarium 1.

§. 21. Cum in Problemate tertio reductio diuisorum ad formam $pp+nqq$ plerumque vnico tantum modo succedat, in casu problematis quarti talis reductio semper infinitis modis succedit; semper enim numeros p et q infinitis modis ita assumere licet, vt vel ipse diuisor D vel Df formulae $pp-nqq$ aequetur.

Corol-

Corollarium 2.

§. 22. Casu autem huius exempli notari meretur, si fuerit $D = pp - 2qq$, tum etiam fore $D = 2rr - ss$, quoniam hae duae formae inter se aequales fieri possunt, iis enim aequatis fit

$$pp + ss = 2(qq + rr) = (q + r)^2 + (q - r)^2,$$

ita vt sit $p = q + r$ et $s = q - r$.

Exemplum 5.

§. 23. *Inueuire diuisores primos formae* pp+3qq.

Solutio.

Quia hic est $n = 3$, ideoque $g < 1$, tantum erit $g = 0$, hincque diuisor $D = rr + 3ss$, ita vt etiam hoc casu omnes diuisores primi sint formae $pp + 3qq$. Quia autem limes pro g inuentus ipsi vnitati aequatur, quam superare non debet, euoluamus etiam casum $g = 1$, vnde fit $fh = 4$ ideoque vel $f = 1$ et $h = 4$, vel $f = 2$ et $h = 2$. Priori casu fit

$$D = rr + 2rs + 4ss = (r + s)^2 + 3ss,$$

quae est ipsa forma proposita. Altero casu fit

$$D = 2rr + 2rs + 2ss,$$

quae forma cum factorem habeat 2 statui debet

$$D = rr + rs + ss$$

quae autem pariter ad propositam reducitur. Nam si s est numerus par, puta $s = 2t$, erit

$$D = rr + 2rt + 4tt = (r + t)^2 + 3tt;$$

sin

sin autem s est numerus impar, etiam r debebit esse impar, quia alioquin ad casum praecedentem reuolueremur; erit ergo $r+s$ numerus par, vnde posito $r=2t-s$ fiet

$$D = 4tt - 2ts + ss = 3tt + (t-s)^2,$$

vnde patet superiorem conclusionem etiamnunc valere, semperque esse $D = rr + 3ss$. Deinde pro formula $12i + a$, ob $a = pp$ erit $a = 1$, tum vero formula $a = pp + 3$ dat $a = 7$, vnde omnes diuisores continebuntur in alterutra harum formularum: $12i + 1$ vel $12i + 7$, quas coniunctim ita repraesentemus: $12i + 1, + 7$, vel etiam hoc modo $12i + 1, - 5$. Si enim omnes valores ipsius a infra $2n$ in genere deprimere liceat, admittendis scilicet numeris negatiuis, tum altera formula $12i = \alpha$, in qua nullus diuisor continetur, erit $12i + 5$ et $12i + 11$, vel $12i - 1, + 5$, vnde patet in genere valores ipsius α negatiuos esse ipsius a.

Exemplum 6.

§. 23. *Inuenire diuisores formulae* $pp - 3qq$ *etiam* $3pp - qq$.

Solutio.

Applicando hic problema quartum erit $n = 3$, ideoque $g < \sqrt{\tfrac{3}{3}}$, consequentur $g = 0$ et $fh = 3$, vnde diuisor erit $D = rr - 3ss$. Hinc patet, hos numeros nullos alios diuisores admittere, nisi qui sint eiusdem formae. Deinde pro formula $12i + a$, ob $a = pp$, vel $a = 3 - pp$, alii valores non prodeunt, praeter $a = 1$, ita vt omnes diuisores contineantur in hac forma: $12i \pm 1$. Formula igitur diuisores excludens erit $12i \pm 5$.

Scho-

Scholion.

§. 24. Istas formulas iam olim expediui, et demonstraui eas alios diuisores non admittere, nisi qui sint eiusdem formae, id quod in maioribus numeris pro n assumtis non semper contingit. Conueniet autem eos casus excludere, quibus n est vel numerus quadratus, vel per quadratum diuisibilis. Si enim foret $n = kmm$, tum formula $pp \pm kmmqq$ conueniret cum hac: $pp \pm kqq$.

Exemplum 7.

§. 25. *Inuenire diuisores numerorum* $pp + 5qq$.

Solutio.

Ob $\sqrt{\tfrac{5}{3}} > g$ erit vel $g = 0$ vel $g = 1$; priori casu fit $fh = 5$, posteriori vero $fh = 6$. Prior casus dat diuisorem $D = rr + 5ss$, quae est ipsa forma proposita; posterior vero dat vel

$$D = rr + 2rs + 6ss \text{ vel}$$
$$D = 2rr + 2rs + 3ss,$$

quarum formarum illa per reductionem ad primam redit, cum sit

$$D = (r + s)^2 + 5ss,$$

haec vero ab illa discrepat, cum inde fiat

$$2D = 4rr + 4rs + 6ss = (2r + s)^2 + 5ss;$$

vnde patet, omnes diuisores vel ipsos esse numeros huius formae, vel eorum dupla, ita vt, si ipse diuisor D non fuerit formae $pp + 5qq$, eius duplum 2D certe futurum sit huius formae. Deinde pro forma $20i + a$, ob $a = pp$ eius valores hinc nati erunt 1 et 9, ex altera autem formula $= pp + 5$ colliguntur iidem valores 1 et 9. Quia vero hic tantum de diuisoribus agitur, pro a etiam sumi poterit $\frac{pp+5}{2}$, vnde oriuntur valores 3, 7, sicque formula omnes diuisores continens erit $20i + 1, + 3, + 7, + 9$, contra vero formula diuisores excludens erit $20i - 1, - 3, - 7, - 9$. Si iam demonstrari posset, istam postremam formulam continere omnes numeros primos, qui nequeunt esse diuisores formae propositae, simuldemonstratum foret, omnes numeros primos in priore forma contentos certo esse diuisores cuiuspiam numeri formae $pp + 5qq$, ideoque vel ipsos vel eorum dupla eandem formam habere debere. Tales autem numeri vsque ad centum sunt

1, 3, 7, 23, 29, 41, 43, 47, 61, 67, 83, 89.

Exemplum 8.

§. 26. *Inuenire diuisores numerorum formae* $pp - 5qq$.

Solutio.

Hic ex problemate quarto est $n = 5$, vnde ob $g < \sqrt{\frac{n}{3}}$ sumi poterit $g = 0$, vel etiam $g = 1$. Nihil enim nocet sumere $g = 1$; superfluum tantum foret, ipsi maiorem valorem tribuere. At $g = 0$ dat diuisorem $rr - 5ss$, hoc est formae propositae; alter vero valor $g = 1$ dat $fh = 4$ ideoque vel

$$D = rr + 2rs - 4ss, \text{ vel}$$

$$D = 2rr + 2rs - 2ss.$$

Prior reducitur ad $D = (r+s)^2 - 5ss$, hoc est ad propositam, posterior vero per 2 diuisa dat diuisorem,

$$D = rr + rs - ss,$$

quae

quae forma etiam ad propositam reducitur, quod ita ostendo. Vel ambo numeri r et s erunt impares, vel altera par, alter impar. Pro casu posteriore sit $s = 2t$ eritque

$$D = rr + 2rt - 4tt, \text{ siue } D = (r+t)^2 - 5tt.$$

Sin autem ambo numeri sint impares, erit eorum summa $r + s$ par, puta $2t$, ideoque $r = 2t - s$, vnde fit

$$D = 4tt - 2ts - ss = 5tt - (t+s)^2.$$

Patet igitur, omnes diuisores numerorum formae propositae quoque eiusdem esse formae. Iam pro forma $20i \pm a$ valor $a = pp$ praebet 1 et 9, alter autem valor $a = 5 - pp$ praebet itidem 1 et 9, ita vt omnes diuisores contineantur in hac forma $20i \pm 1, \pm 9$. Altera autem forma diuisores excludens erit $20i \pm 3, \pm 7$.

Scholion.

§. 27. Quoniam ex his exemplis iam satis liquet, quomodo pro minoribus numeris n singulas has operationes institui oporteat, aliquot exempla circa numeros maiores adhuc afferamus.

Exemplum 9.

§. 28. *Inuenire omnes diuisores primos numerorum formae $pp + 17qq$.*

Solutio.

Cum sit $\sqrt{\tfrac{17}{3}} < 3$, pro g habebimus tres valores 0, 1, 2. Primo sit $g = 0$, ideoque $fh = 17$, hinc diuisor oritur $G = rr + 17ss$, ideoque ipsius formae propositae. Secundo sumatur $g = 1$, erit $fh = 18 = 1.18 = 2.9 = 3.6$, vnde nascuntur hae formae:

1°. D

1°. $D = rr + 2rs + 18ss = (r+s)^2 + 17ss$

2°. $D = 2rr + 2rs + 9ss$, vnde fit

$2D = 4rr + 4rs + 18ss = (2r+s)^2 + 17ss$;

ita vt $2D$ sit formae propositae:

3°. $D = 3rr + 2rs + 6ss$

cuius triplum induit formam propositam:

Tertio sit $g = 2$ ideoque $fh = 21 = 1.\ 21 = 3.7$; vnde oritur

1°. $D = rr + 4rs + 21ss = (r+2s)^2 + 17ss$

2°. $D = 3rr + 4rs + 7ss$

cuius triplum iterum formam propositam induit. Quamobrem omnes diuisores ita erunt comparati, vt vel ipsi, vel eorum dupla, vel eorum tripla habeant formam propositam. Quod deinde ad formam $68i + a$ attinet, valor $a = pp$ praebet numeros, 1, 9, 25, 49, 13, 53, 33, 21; alter autem valor $a = pp + 17$ dat 21, 33, qui numeri cum praecedentibus conueniunt. Qui autem hic etiam subdupla et subtripla occurrere possunt, primo patet formam $a = \frac{pp}{2}$ nullos dare valores idoneos: at $a = \frac{pp}{3}$ sequentes praebet numeros: 3, 27, 7, 11, 39, 23, 31, 63. Deinde vero formula $a = \frac{pp+17}{2}$ dat 9, 21, etc. qui numeri iam occurrunt. Denique formula $a = \frac{pp+17}{3}$ praebet 7, 11, 27, etc. qui itidem iam adsunt. Quamobrem omnes valores idonei pro a erunt

1, 3, 7, 9, 11, 13. 21, 23, 25, 27, 31, 33, 39, 49, 53, 63.

Hi autem numeri multo facilius inueniri possunt; statim enim atque aliquos tantum reperimus, quoniam nouimus, eorum producta ex binis pluribusue etiam occurrere debere, ante omnia autem omnes numeri quadrati per se occurrunt, ex ex quibus, quia etiam 3 occurrit, iam omnes plane reperiuntur. Quod si iam omnes hos numeros infra semissem numeri 68 deprimamus, dum maiorum complementa ad 68 signo — affecta apponimus, tum valores ipsius a sequentem seriem constituent:

$$+1, +3, -5, +7, +9, +11, +13, -15, -19, +21, +23, +25, +27, -29, +31, +33.$$

Si iam omnium horum numerorum signa mutemus, obtinebimus omnes valores litterae α, pro formula $68i+\alpha$, ex qua omnes diuisores sunt exclusi.

Corollarium 3.

§. 29. Hinc igitur perspicuum est, etiam pro omnibus aliis numeris positiuis loco n assumtis in valoribus litterae a omnes plane occurrere numeros impares minores quam $2n$, et qui simul ad n sint primi, dum alii signo $+$, alii signo $-$ sunt affecti.

Exemplum 10.

§. 10. *Inuenire omnes diuisores primos numerorum in hac formula:* $pp-19qq$, *vel etiam* $19pp-qq$ *contentorum.*

Solutio.

Hic igitur ob $n=19$ erit $g<\sqrt{\frac{19}{3}}$, ideoque $g<2$, vnde habebimus vel $g=0$ vel $g=1$. Sit primo $g=0$ eritque diuisor $D=rr-19ss$ ob $fh=19$, ideoque hi diuisores iam sunt ipsius formae propositae. Sit porro $g=1$ fietque

$$fh = 19 - 1 = 18 = 1 . 18 = 2 . 9 = 3 . 6;$$

vnde tres caſus ſunt euoluendi:

1°. $D = rr + 2rs - 18ss = (r+s)^2 - 19ss$

quae forma iam in propoſita continetur.

2°. $D = 2rr + 2rs - 9ss,$

cuius duplum ad formam propoſitam redit,

3°. $D = 3rr + 2rs - 6ss$

cuius triplum in forma propoſita continetur. Sicque omnes diuiſores quaeſiti vel ipſi, vel eorum dupla, vel eorum tripla in forma propoſita continentur. Deinde pro forma $4ni + a$, ſiue $76i + a$ valores ipſius a ex ſequentibus formulis deriuari debent:

1°. $a = pp$ dat 1, 9, 25, 49, 5, 45, 17, 73, 61.

2°. $a = \frac{pp}{2}$ dat nullos valores idoneos, quia omnes forent pares. 3°. $a = \frac{pp}{3}$, ſiue $a = 3tt$, praebet hos valores:

3, 27, 75, 71, 15, 59, 51, 67, 31.

4°. $a = 19 - pp$ dat 15, 3, etc.

qui iam occurrunt,

5°. $a = \frac{19 - pp}{2}$ dat 9, 5, 3, etc.

qui itidem iam adſunt.

6°. $a = \frac{19 - pp}{3}$ dat 5, 1, 15, etc.

qui etiam adſunt. Quamobrem omnes numeri idonei pro a aſſumendi, quoniam tam poſitiue quam negatiue accipi poſſunt, infra 38 deprimi poſſunt, dum ſcilicet maiorum complementa ad 76 apponuntur.

1, 3, 5, 9, 15, 17, 25, 27, 31.

Pro

Pro altera autem forma $76i \pm \alpha$, in qua nulli diuisores occurrere possunt, valores ipsius α sunt sequentes:

$$7, 11, 13, 21, 23, 29, 33, 35, 37.$$

Scholion.

§. 31. Hactenus alios numeros pro n non assumsimus, praeter primos, quamobrem etiam adhuc adiungamus duo exempla circa numeros compositos.

Exemplum II.

§. 32. *Inuenire omnes diuisores primos numerorum in hac forma contentorum:* $pp+30qq$.

Solutio.

Hic ob $n=30$ et $g<\sqrt{10}$, loco g quatuor valores assumi conueniet, 0, 1, 2, 3 quos ergo singulos percurramus

I. $g=0$ praebet $fh=30$, vnde pro diuisore D sequentes formulae nascuntur:

1°. $D = rr + 30ss$,

2°. $D = 2rr + 15ss$;

3°. $D = 3rr + 10ss$

4°. $D = 5rr + 6ss$

quarum prima cum forma proposita congruit, tum vero secundae duplum, tertiae triplum et quartae quintuplum; vbi notetur, loco quintupli etiam sextuplum sumi posse, quandoquidem si fuerit

$$5D = pp + 30qq,$$

tum

tum etiam erit

$$6D = pp + 30qq$$

II. Sit iam $g = 1$, erit $fh = 31$, vnde vnica forma nascitur

$$D = rr + 2rs + 31ss = (r + s)^2 + 30ss ,$$

quae est ipsa forma proposita.

III. Sit $g = 2$, erit $fh = 34 = 1.34 = 2.17$; vnde duae formae nascuntur

$$1^{\circ}.\ D = rr + 4rs + 34ss = (r + 2s)^2 + 30ss.$$

$$2^{\circ}.\ D = 2rr + 4rs + 17ss ,$$

cuius duplum ad formam propositam reducetur.

IV. Sit $g = 3$ eritque $fh = 39 = 1.39 = 3.13$, vnde iterum duae nascuntur formae

$$1.\ D = rr + 6rs + 39ss = (r + 3s)^2 + 30ss$$

$$2^{\circ}.\ D = 3rr + 6rs + 13ss ,$$

cuius triplum induit formam propositam. Ex his igitur sequitur, omnes diuisores D ita esse comparatos, vt vel D, vel 2 D, vel 3 D, vel 6 D in forma proposita contineantur. Deinde vero pro forma $4ni + a = 120i + a$ ante omnia notetur, multitudinem omnium numerorum minorum quam 120 simulque ad 120 primorum esse 32, vnde iam certo inferre possumus, numerum valorum tam litterae a quam α esse 16. Cum igitur primo in a omnes numeri quadrati occurrant, formula $a = pp$ dabit hos tantum numeros: 1 et 49: at vero formae $\frac{pp}{2}$, $\frac{pp}{3}$ et $\frac{pp}{6}$ nullos plane praebent numeros ad 120 primos. Altera vero forma $a = pp + 30$ praebet hos tantum numeros: 31, 79. Hinc autem porro $a = \frac{pp+30}{2}$, siue haec: $a = 2tt + 15$ praebet 17, 23, 47, 113. Porro $a = \frac{pp+30}{3}$, siue $a = 3tt + 10$ praebet 13, 37. Haec igitur

tur forma tantum dat duos valores. Denique $a = \frac{tt+30}{6}$, siue $a = 6tt + 5$ dat hosce: 11, 29, 59, 101. Hoc autem modo tantum 14 prodierunt valores pro littera a, ita vt duo adhuc desiderentur. Verum hic perpendendum est, loco formulae $pp + 30$ generalius poni potuisse $pp + 30qq$, vnde sumendo $p = 3t$ et per 3 diuidendo statui poterit $a = 3tt + 10qq$. Sit nunc $q = 2$, fietque $a = 3tt + 40$, vnde casus $t = 1$ praebet $a = 43$, at $t = 3$ dat $a = 67$; hocque modo nacti sumus omnes 16 valores ipsius a, qui ordine ita procedunt:

1, 11, 13, 17, 23, 29, 31, 37, 43, 47, 49, 59, 67, 79, 101, 113.

Quod si iam loco numerorum maiorum quam 60 eorum complementa ad 120 cum signo − scribantur, isti numeri ita disponi poterunt:

$+1, -7, +11, +17, -19, +23, +29, +31, +37, -41,$
$+43, +47, +49, -50, +59,$

vbi omnes plane numeri impares ad 30 primi occurrunt: vel signo + vel − affecti, vbi si signa mutentur, habebuntur omnes valores litterae α pro formula $120i + \alpha$, cuius omnes numeri ex classe diuisorum excluduntur.

Corollarium 1.

§. 33. Omnes ergo diuisores nnmerorum formae $pp + 30qq$ in quatuor classes distribuuntur, quarum prima continet eos, qui ipsi sunt formae $pp + 30qq$; secunda classis vero eos, quorum dupla sunt eius formae, tertia, quorum tripla et quarta denique eos, quorum quintupla vel etiam sextupla ad formam $pp + 30qq$ reduci possunt. Hae igitur quatuor classes, si formam propositam $pp + 30qq$ littera F, diuisores vero littera D designemus, hoc modo repraesentari possunt:

I. $D = F$; II. $2D = F$; III. $3D = F$; IV. $5D = F$; vbi notaſſe iuuabit, ſi fuerit $2D = F$, tum etiam fore $15D = F$; ſimilique modo ſi fuerit $3D = F$, erit etiam $10D = F$; at ſi fuerit $5D = F$, erit etiam $6D = F$.

Corollarium 2.

§. 34. Quando dicimus, omnes diuiſores numerorum formae propoſitae $pp + 30qq$ in forma $120i + a$ contineri, id non ita eſt intelligendum, quaſi omnes numeri in formula $120i + a$ contenti eſſent diuiſores, ſed inde excludi debent omnes illi, qui per quempiam numerum formae $120i + \alpha$ ſunt diuiſibiles. His autem ſublatis maxime probabile videtur, omnes reliquos numeros formulae $120i + a$, ideoque imprimis numeros primos, certe fore diuiſores cuiuspiam numeri formae $pp + 30qq$. Iſti autem numeri primi in formula $120i + a$ contenti facili negotio quousque libuerit aſſignari poſſunt, quippe qui hoc ordine vsque ad 240 progrediuntur

1, 11, 13, 17, 23, 29, 31, 37, 43, 47, 59, 67, 79, 101, 113, 131, 137, 149, 151, 157, 163, 167, 179, 199, 233.

Corollarium 3.

§. 35. Quoniam omnes diuiſores ſunt quadruplicis generis, inde etiam valores ipſius a in quatuor claſſes diſtribui conueniet, prouti inde oriuntur diuiſores vel primae, vel ſecundae, vel tertiae, vel quartae claſſis, quibus ergo ſubſcribamus characteres cuiusque claſſis 1, 2, 3, 6, hoc modo:

1,	11,	13,	17,	23,	29,	31,	37,	43,	47,	49,	59,	67,	79,	101,	113
1,	6,	3,	2,	2,	6,	1,	3,	3,	2,	1,	6,	3,	1,	6,	2.

Hic igitur notari meretur, ſingulas claſſes quater occurrere.

Exem-

Exemplum II.

§. 36. *Inuenire omnes diuisores primos numerorum in hac forma:* pp − 30qq, *siue in hac:* 30pp − qq *contentorum.*

Solutio.

Cum hic sit $\sqrt{\frac{30}{3}} < 3$, pro littera g habemus tantum tres valores 0, 1, 2. Hinc cum sit $fh = 30 - gg$, pro primo casu erit $fh = 30$, pro secundo $fh = 29$ et pro tertio $fh = 26$, quos igitur casus euoluamus.

I. Sit $g = 0$ et hinc nascuntur sequentes valores:

1°. $D = rr - 30ss$,

2°. $D = 2rr - 15ss$,

3°. $D = 3rr - 10ss$,

4°. $D = 5rr - 6ss$.

II. Si $g = 1$ vnica forma nascitur

$$D = rr + 2rs - 29ss = (r+s)^2 - 30ss,$$

quae ergo est ipsa forma proposita.

III. Si $g = 2$ oriuntur duae formulae

1°. $D = rr + 4rs - 26ss = (r+2s)^2 - 30ss$:

iterum ipsa proposita;

2°. $D = 2rr + 4rs - 13ss$,

cuius duplum fit numerus formae propositae. Hinc ergo nascuntur quadruplicis generis diuisores, qui posita littera F pro formula proposita sunt

I. $D = E$. II. $2D = F$. III. $3D = F$. IV. $6D = F$.

Deinde vero pro formula omnes diuisores continente $120i + a$ erit primo vel $a = pp$, vel $a = \frac{pp}{2}$ vel $a = \frac{pp}{3}$ vel $a = \frac{pp}{6}$, vnde

vnde alii numeri ad 30 primi oriri nequeunt nisi ex prima forma $a = pp$, ideoque duo tantum valores hinc nascuntur: scilicet 1 et 49. Altera autem forma erat $a =$ vel $30 - pp$, vel $\frac{30 - pp}{2}$, vel $\frac{30 - pp}{3}$, vel $\frac{30 - pp}{5}$, quarum prima $a = 30 - pp$ praebet hos numeros: 29, 19, 91. Quia autem loco 30 ponere possumus $30qq$, formula $a = 120 - pp$ praebet insuper hos valores: 119, 71. Secunda ad formam $a = 2tt - 15$ reducta dat hosce numeros: 13, 7, 17, 83, 113, 107. Huic vero formulae aequiualet $15pp - 2qq$, ergo sumto $p = 3$ erit quoque $a = 135 - 2qq$, vnde prodeunt 13, 7, 103; ideoque insuper nouus 103 accedit. Ex tertia forma $a = 3tt - 10$ hosce nouos numeros nanciscimur: 7, 17, forma autem affinis $a = 10tt - 3$ praebet insuper 37. Ex vltima forma $5tt - 6$ nascuntur hi valores: 1, 119; ex forma vero affini $a = 6tt - 5$ isti: 1, 19, 49, 91. Hinc imprimis notandum est, eosdem numeros ex diuersis classibus oriri posse. Prodierunt autem hactenus

1, 7, 13, 17, 19, 29, 37, 49, 71, 83, 91, 103, 107, 113, 119

quorum valorum numerus quidem tantum est 15, cum esse deberet 16; quia autem nouimus, cuiusque numeri complementum ad 120 etiam occurrere debere, iste defectus facile suppletur. Deerat scilicet 101 tanquam complementum ipsius 19. Quia autem numeri a tam positiue quam negatiue accipi possunt, complementa reiicere licet, ita vt pro a habeamus octo sequentes valores:

1, 7, 13, 17, 19, 29, 37, 49

reliqui igitur numeri praebent valores litterae α, qui erunt totidem

11, 23, 31, 41, 43, 47, 53, 59.

Corol-

Corollarium 1.

§. 37. Hoc igitur casu, admisso meo Theoremate, quod omnes numeri primi in forma $4ni + a$ contenti simul sint diuisores numerorum formae $pp \pm nqq$; numeri primi ex nostra formula $120i \pm a$ orti vsque ad 240 sunt sequentes:

1, 7, 13, 17, 19, 29, 37, 71, 83, 101, 103, 107, 113, 127, 137, 139, 149, 157, 191, 223, 227, 233.

Corollarium 2.

§. 38. Quoniam in hac euolutione vidimus, eosdem numeros ex diuersis classibus ortos esse, manifestum est nequicquam quatuor classes diuersas esse constitutas, sed binas earum in vnam coalescere posse. Primo enim omnes diuisores quartae classis, pro quibus erat $5D = F$, siue etiam $6D = F$, iam in prima classe $D = F$ reperiuntur, ita vt semper, quoties fuerit $5D = F$, etiam futurum sit $D = F$. Simili modo diuisores tertiae classis etiam continentur in classe secunda. Quod si enim fuerit $3D = F$, semper etiam erit $2D = F$, quamobrem omnes diuisores pro forma proposita $pp - 30qq$, vel $30pp - qq$ ad duas tantum classes priores reuocari possunt: semper enim erit vel $D = F$ vel $2D = F$.

Corollarium 3.

§. 39. Omnes igitur numeri primi ex nostra forma $120i \pm a$ oriundi duplicis erunt generis, dum vel ipsi vel eorum dupla formam propositam haberi possunt, quos simili modo vt ante distingui conueniet, subscribendo singulis valoribus characteres vel 1 vel 2

1, 7, 13, 17, 19, 29, 37, 49
1, 2, 2, 2, 1, 1, 2, 1.

Vbi notetur, ambos characteres totidem occurrere.

Scholion.

§. 40. Quod si ergo pro numeris cuiuscunque formae $pp \pm nqq$ omnes diuisores primi desiderentur, eos facillime ex nostra forma generali $4ni + a$ assignare licet, dum contra, si formulis ab Illustri *La Grange* exhibitis vti vellemus, opus foret maxime molestum, ex singulis formis $frr + 29rs + hss$ omnes numeros primos elicere; quamobrem maxime est optandum, vt demonstratio firma illius mei asserti detegatur, quippe quo demum ista Theoria ad summum perfectionis gradum euehetur. Arbitror autem, talem demonstrationem mox fortasse sperari posse, si sequentia momenta probe perpendantur.

1). Postquam pro formula proposita quacunque $pp \pm nqq$ ambae meae formulae $4ni + a$ et $4ni + \alpha$ fuerint constitutae, eae simul omnes plane numeros impares ad propositum n primos complectuntur; tum vero omnes diuisores ad formam priorem $4ni + a$ referuntur; nulli autem numeri alterius formae $4ni + \alpha$ possunt esse diuisores propositae, siue omnes numeri posterioris formae ex classe diuisorum penitus excluduntur.

2). Probe perpendatur, quouis casu omnes valores ipsius a egregia lege inter se cohaerere, ita vt omnes coniunctim quasi ambitum quendam completum constituant, in quo nihil deficiat nihilque abundet, quandoquidem omnia producta ex binis pluribusue horum numerorum iterum in

eadem

eadem classe occurrunt, ita vt simul atque aliqui valores idonei pro a fuerint inuenti, ex iis reliqui omnes facile definiri queant, praecipue quoniam omnes numeri quadrati eorumue residua respectu diuisoris $4n$ certe ingrediuntur. Vnde si hoc modo omnia producta, atque etiam potestates numerorum iam inuentorum inserantur, mox tota ista classis ita adimplebitur, vt multitudo omnium numerorum huc pertinentium semper sit semissis omnium plane numerorum ad $4n$ primorum eoque minorum; altera vero semissis praebebit classem numerorum α, qui nullo modo diuisores euadere possunt.

3). Hinc igitur patet, ambas istas classes discrimine maxime memorabili et in natura ipsa numerorum fundato a se inuicem discrepare, atque adeo essentialiter a se inuicem distingui, ita vt numeri alterius classis natura sua ab altera classe prorsus sint diuersi.

4). Quoniam nulli numeri classis $4ni + \alpha$ vnquam esse possunt diuisores vllius numeri formae $pp \pm nqq$: ista classis tanquam origo spectari debet omnium numerorum, quorum natura ab indole diuisorum abhorret, quae repugnantia quoque ad omnes numeros extendi debet, qui diuisibiles sunt per vllum numerum classis $4ni + \alpha$. Si enim tales numeri possent esse diuisores, etiam isti huius classis numeri forent diuisores, id quod naturae rei repugnat.

5). Cum autem producta ex binis numeris classis $4ni + \alpha$ in classem diuisorum $4ni + a$ transeant, manifestum est, in prima classe plurimos occurrere debere ab indole diuisorum alienas; omnes scilicet eos, qui per vllum numerum alterius classis sunt diuisibiles.

 6).

6). Quod si iam omnes isti numeri in classe $4ni+a$ deleantur siue excludantur, qui natura diuisorum refragantur, maxime probabile videtur, reliquos numeros omnes indole diuisorum fore praeditos. Cum hoc modo tantum numeri compositi expungantur, euidens est omnes plane numeros primos in forma $4ni+a$, contentos ~~reuera~~ fore diuisores cuiuspiam numeri formae $pp+nqq$. Totum ergo negotium huc redit, vt isti probabilitati vis perfectae demonstrationis concilietur. Haec autem veritas si qua est elegantius ita proponi potest.

Theorema demonstrandum.

§. 41. Si fuerit a diuisor cuiuspiam numeri formae $pp+nqq$, ita vt sit $a\mathrm{D}=pp+nqq$, tum quoties $4ni+a$ est numerus primus, toties quoque erit $\mathrm{D}(4ni+a)$, numerus formae $pp+nqq$. Hic autem sequentia notari oportet. 1) Numeros p et q inter se esse debere primos. 2) Diuisorem a etiam primum esse debere ad n, quoniam diuisores ipsius n hinc excluduntur. 3) Quod si forte eueniat, vt numerus $\mathrm{D}(4ni+a)$ non videatur in forma $pp+nqq$ contineri, tum semper eius quadruplum, vel etiam eius productum per aliud quadratum, certe in ea contineri. Quoniam igitur hoc casu erit

$$\mathrm{D}(4ni+a)=(\tfrac{p}{2})^2+n(\tfrac{q}{2})^2,$$

haec resolutio nullam exceptionem mereri est censenda. Ita cum sit $27=4^2+11.1^2$, erit $a=27$ et $n=11$ et $\mathrm{D}=1$, vnde formula $4ni+a$ euadit $44i+27$, quae casu $i=1$ praebet 71, hoc est numerum primum; neque tamen in integris esse potest $71=pp+11qq$. Est vero

$$4 \cdot 71 = 284 = 3^2 + 11 \cdot 5^2 \text{ ideoque}$$

$$71 = (\tfrac{3}{2})^2 + 11 \cdot (\tfrac{5}{2})^2.$$

Tales autem casus raro occurrunt et ideo non sunt excipiendi, quia numeri formulae $4ni + \alpha$ ita ex classe diuisorum excluduntur, vt, etiamsi pro p et q numeri fracti accipiantur, tamen nunquam diuisores esse queant.

Scholion.

§. 42. Superfluum foret has inuestigationes ad huiusmodi formulas: $mpp \pm nqq$ extendere, cum omnes diuisores numerorum formae $mpp \pm nqq$ semper sint etiam diuisores numerorum formae $pp \pm mnqq$. Quae igitur olim in Tomo XIV Comment. Vet. Academiae de diuisoribus numerorum formae $mpp \pm nqq$, sum commentatus et magnam partem ex sola inductione conclusi, nunc per egregias proprietates ab illustri *La Grange* demonstratas non solum plurimum illustrantur, sed etiam ad multo maiorem certitudinis gradum perducuntur, ita vt iam nihil amplius desideretur, nisi vt solida demonstratio Theorematis allati detegatur, quam nunc quidem mox exspectare licebit. Mea autem methodus imprimis hac gaudet praerogatiua, quod eius ope omnes plane diuisores huiusmodi formularum $mpp \pm nqq$ assignari et, quousque libuerit, continuari possunt, id quod insuper sequenti exemplo declarabo.

Exemplum.

§. 43. *Inuenire omnes diuisores formae* $pp + 39qq$.

Primo igitur per formulas illustris *La Grange* quaeramus omnes diuersas formas horum diuisorum, et cum sit

$n = 39$ ideoque $\sqrt{\frac{n}{3}} < 4$, sufficiet pro g assumere hos quatuor valores: 0, 1, 2, 3.

I. Valor igitur $g = 0$ praebet $fh = 39$, vnde hae duae formae nascuntur:

$$1^\circ.\ rr + 39ss, \quad 2^\circ.\ 3rr + 13ss,$$

quarum prior dat diuisores $D = F$ et altera $3D = F$; denotante F formam propositam.

II. Valor $g = 1$ dat $fh = 40$, vnde nascuntur istae formae:

$$1^\circ.\ D = rr + 2rs + 40ss = (r + s)^2 + 39ss,$$

ideoque $D = F$.

$$2^\circ.\ D = 2rr + 2rs + 20ss,$$

quae forma autem numerus primus esse nequit.

$$3^\circ.\ D = 4rr + 2rs + 10ss,$$

quae forma itidem non dat numeros primos.

$$4^\circ.\ D = 5rr + 2rs + 8ss,$$

vnde fit $5D = F$, vel etiam $8D = F$.

III. Casus $g = 2$ dat $fh = 43$, vnde vnica forma oritur

$$D = rr + 4rs + 43ss = (r + 2s)^2 + 39ss$$

ideoque $D = F$.

IV. Casus denique $g = 3$ praebet $fh = 48$, vnde sequentes formae numeros primos continentes oriuntur:

$$1^\circ.\ D = rr + 6rs + 48ss = (r + 3s)^2 + 39ss,$$

ideoque $D = F$.

$$2^\circ.\ D = 3rr + 6rs + 16ss$$

dat

dat $3D = F$, vel etiam $16D = F$. Hinc igitur patet, omnino dari tria genera diuiſorum:

$$1)\ D = F;\quad 2)\ 3D = F,\quad 5D = F.$$

Quibus conſtitutis euoluamus formulam $4ni + a = 156i + a$, vbi primo notetur, omnium numerorum ad 156 primorum ipſoque minorum multitudinem eſſe 48, vnde vsque ad ſemiſſem 78 erunt 24, quorum ſinguli vel poſitiue vel negatiue ſumti praebent valores pro littera a. Iſti ergo numeri erunt.

$$1, 5, 7, 11, 17, 19, 23, 25, 29, 31, 35, 37, 51, 43,$$
$$47, 49, 53, 55, 59, 61, 67, 71, 73, 77$$

vbi primo quadrata habent ſignum $+$, qui ergo ſunt

$$+1, +25, +49;$$

reliquorum vero numerorum quadrata diuiſione per 156 deprimantur infra 78, vnde fiet

$$11^2 = -35,\ 17^2 = -23,\ 19 = +49,\ 23^2 = +61.$$

Pro reliquis numeris conſideremus formam $pp + 39$, vnde ſumto $p = 1$ prodit 40, cuius numeri ad genus tertium pertinentis diuiſor 5 habet ſignum $+$. Iam quia praecedentes numeri ad genus primum ſunt referendi, eorum producta per 5 etiam ad genus tertium referri debebunt, vnde naſcentur ſequentes numeri:

$$+5, +41, -31, -19, -67, -7.$$

Sit nunc $p = 2$ eritque $4 + 39 = 43$, qui eſt diuiſor primae claſſis, vnde etiam numeri huius claſſis iam inuenti per 43 multiplicati dabunt diuiſores primae claſſis, qui autem, cum numerus 43 ſit nimis magnus, facilius ex ſequentibus reperientur. Sumatur igitur $p = 3$ eritque $pp + 39 = 48$, cuius diuiſor 3 iam eſt excluſus. Sit ergo $p = 4$, erit-

que

que $16 + 39 = 55$, cuius diuisorem 5 iam tractauimus; alter vero diuisor 11 etiam ad tertiam classem pertinet, per hunc ergo numeri primae classis multiplicati erunt

$$+11, +59, -37, -73, +71, +47.$$

Multiplicentur etiam numeri tertiae classis per 11 et producta depressa, qui sunt

$$+55, -17, -29, -53, +43, -77.$$

reuertentur ad classem primam. Hoc modo omnes nostri numeri signa sua debita sunt adepti, qui cum vel ad primam vel ad tertiam classem referantur, manifestum est nullos diuisores secundae classis relinqui. Omnes scilicet hi numeri iam in prima classe continentur, quamobrem omnes valores ipsius a cum characteribus suis I vel III subscriptis ita se habebunt:

+1,	+5,	−7,	+11,	−17,	−19,	−23,	+25,	−29,	−31,	35,	−37
I	III	III	III	I	III	I	I	I	III	I	III

+41,	+43,	+47,	+49,	−53,	+55,	+59,	+61,	−67,	+71,	−73,	−77
III	I	III	I	I	I	III	I	III	III	III	I.

Neque vero classis secunda prorsus est inutilis: dantur enim numeri primi, quos ad primam classem retulimus, quorum resolutio in integris non succedit, atque adeo denominatorem quadratum 16 postulat, cuiusmodi numerus est 61, qui aliter ad primam classem redigi nequit, nisi hoc modo: $61 = (\frac{25}{4})^2 + 39(\frac{3}{4})^2$. Est vero $3.61 = 183 = 12^2 + 39.1^2$. Quod si iam valores negatiui pro a inuenti in positiuos conuertantur, sumendis complementis ad 156, sequentes valores prodibunt:

1,	5,	11,	25,	41,	43,	47,	49,	55,	59,	61,	71,	79,	83,	89,	113,
I	III	III	I	III	I	III	I	I	III	I	III	I	III	III	I

119,	121,	125,	127,	133,	137,	139,	149
III	I	III	I	I	III	I	III.

Nunc

Nunc igitur omnes numeri primi in forma $156i + a$ contenti certe erunt diuisores cuiuspiam numeri formae $pp+39qq$, atque adeo vel ipsi, vel eorum quintupla, vel etiam tripla erunt numeri huius formae. Hinc ergo omnes diuisores primi ab 1 vsque ad 312 erunt sequentes:

1, 5, 11, 41, 43, 47, 59, 61, 71, 79, 83, 89, 113, 127, 137, 139, 149, 157, 167, 181, 197, 199, 211, 227, 239, 269, 277, 281, 283, 293.

Corollarium 1.

§. 44. Quo facilius intelligatur, cur hoc casu classis secunda ad primam sit reuoluta, iam supra ostendimus, si diuisor fuerit

$$D = frr + 2grs + hss,$$

existente $fh = gg + n$, tum non solum Df, sed etiam Dh ad formam $pp + nqq$ reduci posse. Hinc autem generalius si fuerit

$$k = ftt + 2gtu + huu,$$

tum productum Dk etiam erit numerus formae $pp+nqq$; facto enim calculo reperitur

$$Dk = (frt+g(ts+ru)+hsu)^2 + n(ts-ru)^2.$$

Quod si ergo k fieri queat quadratum, vel diuisibile per quadratum, tum hoc quadratum omitti poterit. Nam si fuerit $Dkll$ numerus formae $pp + nqq$, tum, etiam admissis fractionibus, erit quoque Dk eiusdem formae. Ita nostro casu pro diuisoribus secundae classis erat

$$D = 3rr + 13ss,$$

ideoque $k = 3tt + 13uu$, cuius valor sumto $t = 1$ et $u = 1$ fiet $k = 16$, qui cum sit numerus quadratus, haec forma ad primam reducitur.

Corollarium 2.

§. 45. Nunc igitur omnia Theoremata, quae circa huiusmodi diuisores olim in Comment. veter. Tomo XIV. dederam, multo maiorem gradum certitudinis sunt adepta, postquam a celeb. *la Grange* formae istorum diuisorum sunt demonstratae; atque nullum dubium esse videtur, quin mox quod in hoc genere adhuc desideratur perfecta Demonstratione muniatur.

Corollarium 3.

§. 46. Antequam hoc argumentam penitus deseram, memorabilem adhuc obseruationem adiungam circa signa numerorum a, dum scilicet omnes eius valores infra $2n$ deprimuntur. Cum enim horum numerorum primus et vltimus simul sumti fiant $2n$, dispiciendum est, vtrum hi duo numeri habeant vel paria signa vel disparia, vtroque enim casu bini quicunque horum numerorum ab extremis aequidistantes, quorum ergo summa semper est $2n$, etiam habebunt siue eadem signa siue contraria. Ita nostro casu, quo erat $2n = 78$, vltimus 77 habebat signum $-$, dum primus 1 semper habet signum $+$, vnde etiam signa binorum ab extremis aeque distantium perpetuo erunt contraria. E contrario autem in Exemplo 11, vbi erat $2n = 60$, vltimus numerus 59 habebat signum $+$, vnde etiam bini quicunque alii ab extremis aequidistantes eodem signo affecti deprehenduntur, cuius quidem phaenomeni ratio haud difficulter inuestigari poterit. Huiusmodi autem obseruationes laborem inuestigationis Diuisorum non mediocriter subleuant.

SO-

SOLVTIO QVAESTIONIS AD CALCVLVM PROBABILITATIS PERTINENTIS.

QVANTVM DVO CONIVGES PERSOLVERE DEBEANT, VT SVIS HAEREDIBVS POST VTRIVSQVE MORTEM CERTA ARGENTI SVMMA PERSOLVATVR.

§. 1.

Assumimus hic eiusmodi aerarium publicum esse constitutum, cuius facultates quotannis vicesima sui parte augeri queant, ita vt summa 100 Rubellonum post annum ad 105 Rub. excrescat; quare si breuitatis gratia ponamus $\frac{105}{100} = \lambda$, praesens pecuniae summa $= C$ post n annos aestimanda erit $\lambda^n C$. Vicissim autem quaeuis pecuniae summa C post n annos soluenda praesenti tempore valorem habere censenda est $= \frac{C}{\lambda^n}$.

§. 2. Ponamus nunc argenti summam, quam ambo coniuges post vtriusque mortem acquirere optant, esse $= 1000$ Rub. vnde intelligitur si tempus huius solutionis esset cognitum, annorum praeterlapsorum numero existente $= n$, eius valorem praesentem futurum esse $= \frac{1000}{\lambda^n}$. Tantum igitur illi coniuges praesenti tempore in aerarium conferre tenerentur. Verum cum tempus solutionis maxime sit incertum; siquidem demum post vtriusque mortem fieri debet, verum et praesentem valorem huius summae secundum regulas calculi probabilium ex longaeuis mortalitatis obseruationibus petitas determinari oportet. Hunc in finem vtar tabula, quam olim in Tomo Memor. Berol. pro Anno 1760 inserui, vbi, si praemagnus numerus M infantum simul natorum consideretur, eorum numerum post n annos adhuc superstitum indicaui charactere (n) M; ex quo intelligitur, talem characterem (n) designare fractiones eo minores, quo maior fuerit annorum numerus n, ac tandem circa 100 annos prorsus in nihilum abire; Tabulam igitur horum valorum pro singulis annis elapsis hic exponamus.

(1) =

(1) = 0,804	(25) = 0,552	(49) = 0,370	(73) = 0,145
(2) = 0,786	(26) = 0,544	(50) = 0,362	(74) = 0,135
(3) = 0,736	(27) = 0,535	(51) = 0,354	(75) = 0,125
(4) = 0,709	(28) = 0,525	(52) = 0,345	(76) = 0,114
(5) = 0,688	(29) = 0,516	(53) = 0,336	(77) = 0,104
(6) = 0,676	(30) = 0,507	(54) = 0,327	(78) = 0,093
(7) = 0,664	(31) = 0,499	(55) = 0,319	(79) = 0,082
(8) = 0,653	(32) = 0,490	(56) = 0,310	(80) = 0,072
(9) = 0,646	(33) = 0,482	(57) = 0,301	(81) = 0,063
(10) = 0,639	(34) = 0,475	(58) = 0,291	(82) = 0,054
(11) = 0,633	(35) = 0,468	(59) = 0,282	(83) = 0,046
(12) = 0,627	(36) = 0,461	(60) = 0,273	(84) = 0,039
(13) = 0,621	(37) = 0,454	(61) = 0,264	(85) = 0,032
(14) = 0,616	(38) = 0,446	(62) = 0,254	(86) = 0,026
(15) = 0,611	(39) = 0,439	(63) = 0,245	(87) = 0,020
(16) = 0,606	(40) = 0,432	(64) = 0,235	(88) = 0,015
(17) = 0,601	(41) = 0,426	(65) = 0,225	(89) = 0,011
(18) = 0,596	(42) = 0,420	(66) = 0,215	(90) = 0,008
(19) = 0,590	(43) = 0,413	(67) = 0,205	(91) = 0,006
(20) = 0,584	(44) = 0,406	(68) = 0,195	(92) = 0,004
(21) = 0,577	(45) = 0,403	(69) = 0,185	(93) = 0,003
(22) = 0,571	(46) = 0,393	(70) = 0,175	(94) = 0,002
(23) = 0,565	(47) = 0,386	(71) = 0,165	(95) = 0,001
(24) = 0,559	(48) = 0,378	(72) = 0,155	

§. 3. Ponamus nunc praesenti tempore aetatem mariti esse $= a$ annorum, vxoris vero $= b$ annorum, et quo ratiocinia instituenda clarius percipi queant, fingamus simul ingentem numerum talium coniugum, qui sit N, eiusdem aetatis a desse, qui pariter suis haeredibus post vtriusque mortem summam 1000 Rub. acquirere optent, vnde si summa initio

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perſoluenda ſtatuatur $= x$, aerarium ab his omnibus accipiet ſummam $N x$.

§. 4. Sin autem magis arrideat, vt iſtud pretium x non ſtatim ab initio totum, ſed potius per totam vitam aequaliter diſtributum ſoluatur, calculum noſtrum ad duplicem ſolutionem accommodemus, dum altera ſtatim ab initio ſumma $= x$ in aerarium ſoluitur, altera autem quotannis inſuper quaepiam ſumma $= z$ ſoluitur, quamdiu ſcilicet non ſolum ambo coniuges ſed etiam altervter tantum ſuperſtites fuerint. Solutione autem hoc modo abſoluta ſi quis voluerit totum pretium ſtatim ab initio perſoluere, pro hoc caſu poni oportebit $z = 0$ et littera x quaeſitum pretium indicabit. Sin autem quis maluerit hoc. pretium per totam vitam aequaliter diſtribui, poni oportebit $x = z$, eritque z ſumma ſingulis annis ſoluenda vsque ad mortem vtriusque coniugis.

§. 5. His conſtitutis ſtatim ab initio ab omnibus illis N coniugiis ſoluetur ſumma $= N x$. Nunc videamus, poſtquam elapſi fuerint n anni, quot coniugia adhuc tam integra quam diſſoluta, dum ſcilicet interea altervter fuerit mortuus, ſint ſuperfutura; tum enim a ſingulis iſtis in aerarium ſoluetur ſumma $= z$, cuius valor praeſens aeſtimandus eſt $\frac{z}{\lambda^n}$. Praeterea vero pro quouis anno currente inquirendum eſt, quot coniugia penitus extinguantur: quoties enim hoc euenit, toties eorum haeredibus praemium illud 1000 Rubl. perſolui debet, cuius ergo valor praeſens erit $\frac{1000}{\lambda^n}$. Hoc igitur modo calculum noſtrum proſequi oportet vsque ad ex-

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tremum vitae humanae terminum, et cum omnes tam expenſae quam reditus fuerint ad praeſens tempus reducti, eos inter ſe aequari conueniet, vnde pro lubitu ſiue x ſiue z determinare licebit.

§. 6. His praemiſſis incipiamus ab anno primo, cuius initio adeſſe ponuntur N mariti, omnes eiusdem aetatis $= a$, totidemque vxores eiusdem aetatis $= b$, a quibus aerarium accepit ſummam $= Nx$. Nunc igitur elapſo anno primo ſecundum tabulam ſupra allatam numerus maritorum adhuc ſuperſtitum erit $\frac{(a+1)}{(a)}N$, ideoque numerus interea defunctorum $= \frac{(a)-(a+1)}{(a)}N$. Simili modo numerus vxorum adhuc ſuperſtitum erit $\frac{(b+1)}{(b)}N$, earum autem quae interea ſunt mortuae numerus $\frac{(b)-(b+1)}{(b)}N$. Quia igitur quilibet horum maritorum ſuperſtitum initio habuit coniugem, inſtituatur haec proportio: vti numerus omnium vxorum initio ſe habet ad earum numerum ſuperſtitum, ita numerus virorum elapſo anno ſuperſtitum ad numerum eorum, quorum vxores adhuc erunt ſuperſtites, qui ergo numerus erit $\frac{(a+1)}{(a)} \cdot \frac{(b+1)}{(b)}N$, a quibus ſingulis in aerarium ſoluitur ſumma $= z$, cuius valor praeſens cum ſit $\frac{z}{\lambda}$, hinc orietur valor $= \frac{(a+1)(b+1)}{(a)(b)} \frac{Nz}{\lambda}$. Tum vero numerus eorum maritorum, qui interea vxores amiſerint, erit $\frac{(a+1)}{(a)}(1-\frac{(b+1)}{(b)})N$, qui cum itidem in aerarium ſoluant ſummam z, ea ad initium relata erit $\frac{(a+1)}{(a)}(1-\frac{(b+1)}{(b)})\frac{Nz}{\lambda}$; vnde patet, hunc valorem cum praecedente coniunctum fore $\frac{(a+1)}{(a)}\frac{Nz}{\lambda}$, id quod per ſe eſt manifeſtum, quia quilibet maritus ſuperſtes hanc ſummam z ſoluere tenetur, ſiue eius vxor adhuc viuat ſiue ſecus.

§. 7.

§. 7. Consideremus nunc etiam eos maritos, qui intra hunc annum erunt mortui, quorum numerus est $(1 - \frac{(a+1)}{a})N$; vbi duo casus se offerunt. Alter casus eos spectat maritos, quorum vxores adhuc sunt superstites, quorum numerus per superiorem analogiam inuenitur: vti se habet numerus omnium vxorum initio viuentium ad earum numerum post annum superstitum, ita numerus virorum interea defunctorum ad eorum numerum, quorum vxores adhuc sunt superstites, qui ergo numerus erit $\frac{(b+1)}{(b)}(1 - \frac{(a+1)}{(a)})N$, quae cum singulae etiam in aerarium conferant summam $= z$, eius valor ad initium relatus erit $\frac{(b+1)}{(b)}(1 - \frac{(a+1)}{(a)})\frac{Nz}{\lambda}$, vnde omnes reditus primo anno elapso in aerarium influentes erunt

$$\frac{Nz}{\lambda}\left(\frac{(a+1)}{(a)} + \frac{(b+1)}{(b)} - \frac{(a+1)}{(a)}\frac{(b+1)}{(b)}\right),$$

quae ergo quantitas tribus constat partibus. Primo scilicet valor $\frac{z}{\lambda}$ multiplicatur per numerum maritorum superstitum, qui est $\frac{(a+1)}{(a)}N$, deinde etiam per numerum vxorum superstitum, qui est $\frac{(b+1)}{(b)}N$. Hinc autem aufferri debet numerus coniugiorum adhuc integrorum, quia singula non duo z sed tantum vnum z expendunt.

§. 8. Alter casus eos spectat maritos, quorum vxores non amplius sunt superstites. Ex praecedente autem calculo apparet, numerum eorum maritorum mortuorum, quorum vxores interea quoque sunt defunctae, esse $(1 - \frac{(a+1)}{(a)})(1 - \frac{(b+1)}{(b)})N$. Tot ergo coniugia penitus sunt extincta, quorum igitur haeredibus ex aerario soluendum erit praemium constitutum 1000 Rubl. quod cum statim persolui debeat, nullam vsuram lucrari interea potuit, vnde istae expensae ad initium relatae etiamnunc valebunt

$$1000\,N\left(1 - \frac{(a+1)}{a} - \frac{(b+1)}{(b)} + \frac{(a+1)}{(a)}\frac{(b+1)}{(b)}\right).$$

§. 9.

§. 9. Progrediamur nunc ad annum secundum, cuius initio superstites erant mariti $\frac{(a+1)}{(a)}N$, vxores vero $\frac{(b+1)}{(b)}N$, inter quos subsistent adhuc coniugia integra $\frac{(a+1)}{(a)}\frac{(b+1)}{(b)}N$, soluta vero $\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+1)}{(a)}\frac{(b+1)}{(b)}\right)N$; ita vt numerus coniugiorum penitus extinctorum sit

$$N\left(1-\frac{(a+1)}{(a)}-\frac{(b+1)}{(b)}+\frac{(a+1)}{(a)}\frac{(b+1)}{(b)}\right), \text{ siue}$$

$$N\left(1-\frac{(a+1)}{(a)}\right)\left(1-\frac{(b+1)}{(b)}\right).$$

At vero in fine anni secundi numerus maritorum superstitum adhuc erit $\frac{(a+2)}{(a)}N$, numerus vero eorum qui hoc biennio sunt mortui $=\left(1-\frac{(a+2)}{(a)}\right)N$. Similique modo numerus vxorum adhuc superstitum erit $\frac{(b+2)}{(b)}N$, earum vero quae biennio sunt mortuae $\left(1-\frac{(b+2)}{(b)}\right)N$, vnde numerus coniugiorum hoc biennio extinctorum erit

$$\left(1-\frac{(a+2)}{(a)}\right)\left(1-\frac{(b+2)}{(b)}\right)N.$$

Quare cum numerus coniugiorum primo anno extinctorum fuerit $\left(1-\frac{(a+1)}{(a)}\right)\left(1-\frac{(b+1)}{(b)}\right)N$, numerus eorum quae intra hunc secundum annum sunt extincta erit

$$\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+2)}{(a)}-\frac{(b+2)}{(b)}-\frac{(a+1)}{(a)}\frac{(b+1)}{(b)}+\frac{(a+2)}{(a)}\frac{(b+2)}{(b)}\right)N$$

pro quibus singulis quia persoluitur summa mille Rubell., tota summa ob vsuram ad initium relata valebit

$$\frac{1000N}{\lambda}\left[\frac{(a+1)-(a+2)}{(a)}+\frac{(b+1)-(b+2)}{(b)}-\left(\frac{(a+1)(b+1)-(a+2)(b+2)}{(a)(b)}\right)\right].$$

§. 10. Quia nunc initio secundi anni numerus coniugiorum tam integrorum quam solutorum erat

$$N\left(\frac{(a+1)}{(a)}+\frac{(b+1)}{(b)}-\frac{(a+1)}{(a)}\frac{(b+1)}{(b)}\right),$$

si hinc auferamus numerum coniugiorum hoc anno extinctorum, remanebit numerus eorum qui circa finem secundi anni singuli soluent summam z, quorum ergo numerus erit

$$N\left(\frac{(a+2)}{(a)}+\frac{(b+2)}{(b)}-\frac{(a+2)}{(a)}\frac{(b+2)}{(b)}\right).$$

Toties igitur ab his ſumma z in aerarium infertur, vnde ſotus valor ob vſuram duorum annorum minutus primo inito valebit

$$\frac{Nz}{\lambda^2}\left(\frac{(a+2)}{(a)}+\frac{(b+2)}{(b)}-\frac{(a+2)}{(a)}\frac{(b+2)}{(b)}\right).$$

§. 11. His expoſitis iam ad annum quemcunque tequentem progredi poterimus. Ponamus igitur iam elapios eſſe n annos, hocque tempore numerus maritorum ſuperſtitum erit $\frac{(a+n)}{(a)}N$, ante autem iam defunctorum

$$\left(1-\frac{(a+n)}{(a)}\right)N.$$

Eodemque modo numerus vxorum adhuc ſuperſtitum eſt $\frac{(b+n)}{(b)}N$ demortuarum vero $\left(1-\frac{(b+n)}{(b)}\right)N$, vnde numerus coniugiorum tam integrorum quam ſolutorum hoc tempore erit

$$\left(\frac{(a+n)}{(a)}+\frac{(b+n)}{(b)}-\frac{(a+n)}{(a)}\frac{(b+n)}{(b)}\right)N;$$

at vero numerus coniugiorum toto hoc tempore penitus extinctorum erit

$$\left(1-\frac{(a+n)}{(a)}\right)\left(1-\frac{(b+n)}{(b)}\right)N.$$

§. 12. Iam procedamus ad finem iſtius anni, ac ſimili modo numerus coniugiorum, ſiue integrorum, ſiue ſolutorum nunc erit

$$N\left(\frac{(a+n+1)}{(a)}+\frac{(b+n+1)}{(b)}-\frac{(a+n+1)}{(a)}\frac{(b+n+1)}{(b)}\right)$$

a quibus ſingulis in aerarium perſoluitur ſumma z, cuius valor ad initium translatus eſt $\frac{z}{\lambda^{n+1}}$, vnde tota ſumma circa finem huius anni in aerarium ſoluta pro initio valebit

$$\frac{Nz}{\lambda^{n+1}}\left(\frac{(a+n+1)}{(a)}+\frac{(b+n+1)}{(b)}-\frac{(a+n+1)}{(a)}\frac{(b+n+1)}{(b)}\right).$$

Tum

Tum vero numerus omnium coniugiorum ab ipſo initio vsque ad tempus $n+1$ annorum extinctorum erit

$$N\left(1-\frac{(a+n+1)}{(a)}\right)\left(1-\frac{(b+n+1)}{(b)}\right),$$

quare cum vsque ad initium huius anni iam extincta fuiſſent

$$N\left(1-\frac{(a+n)}{(a)}\right)\left(1-\frac{(b+n)}{(b)}\right)$$

coniugia, numerus eorum quae hoc demum anno ſunt extincta erit

$$N\left(\frac{(a+n)-(a+n+1)}{(a)}+\frac{(b+n)-(b+n+1)}{(b)}-\frac{(a+n)(b+n)-(a+n+1)(b+n+1)}{(a)\,(b)}\right)$$

Quoniam igitur pro his ſingulis expendi debet ſumma 1000 Rbl. valor harum expenſarum ad initium relatus erit

$$\frac{1000\,N}{\lambda^n}\left(\frac{(a+n)-(a+n+1)}{(a)}+\frac{(b+n)-(b+n+1)}{(b)}\right.$$
$$\left.-\frac{(a+n)(b+n)-(a+n+1)(b+n+1)}{(a)\,(b)}\right).$$

§. 13. Colligamus nunc omnes tam reditus ex quantitate z oriundos quam expenſas ex ſolutione illorum 1000 Rubell. ortas, ac primo quidem omnes reditus, qui praeter ſummam principalem Nx in aerarium inferuntur, per ternas ſequentes ſeries expreſſi inueniuntur:

$$Nz\left\{\begin{array}{l}+\dfrac{(a+1)}{\lambda(a)}+\dfrac{(a+2)}{\lambda^2(a)}+\dfrac{(a+3)}{\lambda^3(a)}\cdots\cdots+\dfrac{(a+n)}{\lambda^n(a)}\\[2ex]+\dfrac{(b+1)}{\lambda(b)}+\dfrac{(b+2)}{\lambda^2(b)}+\dfrac{(b+3)}{\lambda^3(b)}\cdots\cdots+\dfrac{(b+n)}{\lambda^n(b)}\\[2ex]-\dfrac{(a+1)(b+1)}{\lambda(a)(b)}-\dfrac{(a+2)(b+2)}{\lambda^2(a)(b)}-\dfrac{(a+3)(b+3)}{\lambda^3(a)(b)}\cdots\cdots-\dfrac{(a+n)(b+n)}{\lambda^n(a)(b)}\end{array}\right\}$$

Quod ſi ergo breuitatis gratia ſtatuamus

$$P = \frac{(a+1)}{\lambda} + \frac{(a+2)}{\lambda^2} + \frac{(a+3)}{\lambda^3} + \frac{(a+4)}{\lambda^4} + \ldots\ldots \frac{(95)}{\lambda^{95-a}}$$

$$Q = \frac{(b+1)}{\lambda} + \frac{(b+2)}{\lambda^2} + \frac{(b+3)}{\lambda^3} + \frac{(b+4)}{\lambda^4} + \ldots\ldots \frac{(95)}{\lambda^{95-b}}$$

$$R = \frac{(a+1)(b+1)}{\lambda} + \frac{(a+2)(b+2)}{\lambda^2} + \frac{(a+3)(b+3)}{\lambda^3} + \text{etc.}$$

erit tota summa redituum

$$N x + N z \left(\frac{P}{(a)} + \frac{Q}{(b)} - \frac{R}{(a)(b)}\right).$$

Colligamus simili modo omnes expensas in vnam summam, quae summa ex sex sequentibus seriebus erit composita:

$$1000\,N \left\{ \begin{array}{lllll} \frac{(a)}{(a)} & + \frac{(a+1)}{\lambda(a)} & + \frac{(a+2)}{\lambda^2(a)} & + \frac{(a+3)}{\lambda^3(a)} & + \text{etc.} \\ - \frac{(a+1)}{(a)} & - \frac{(a+2)}{\lambda(a)} & - \frac{(a+3)}{\lambda^2(a)} & - \frac{(a+4)}{\lambda^3(a)} & - \text{etc.} \\ + \frac{(b)}{(b)} & + \frac{(b+1)}{\lambda(b)} & + \frac{(b+2)}{\lambda^2(b)} & + \frac{(b+3)}{\lambda^3(b)} & + \text{etc.} \\ - \frac{(b+1)}{(b)} & - \frac{(b+2)}{\lambda(b)} & - \frac{(b+3)}{\lambda^2(b)} & - \frac{(b+4)}{\lambda^3(b)} & - \text{etc.} \\ - \frac{(a)(b)}{(a)(b)} & - \frac{(a+1)(b+1)}{\lambda(a)(b)} & - \frac{(a+2)(b+2)}{\lambda^2(a)(b)} & - \frac{(a+3)(b+3)}{\lambda^3(a)(b)} & - \text{etc.} \\ + \frac{(a+1)(b+1)}{(a)(b)} & + \frac{(a+2)(b+2)}{\lambda(a)(b)} & + \frac{(a+3)(b+3)}{\lambda^2(a)(b)} & + \frac{(a+4)(b+4)}{\lambda^3(a)(b)} & + \text{etc.} \end{array} \right\}$$

§. 14. Perspicuum est etiam hic summas trium serierum constitutas P, Q, R, commode in subsidium vocari posse, hincque omnes expensas ad initium relatas expressum iri per sequentem formam:

$$1000\,N \left(\frac{(a)+P}{(a)} - \frac{\lambda P}{(a)} + \frac{(b)+Q}{(b)} - \frac{\lambda Q}{(b)} - \frac{(a)(b)-R}{(a)(b)} + \frac{\lambda R}{(a)(b)}\right), \text{ seu}$$

$$1000\,N \left(1 + \frac{(1-\lambda)P}{(a)} + \frac{(1-\lambda)Q}{(b)} + \frac{(\lambda-1)R}{(a)(b)}\right),$$

consequenter aequatio pro solutione quaestionis propositae erit

$$x + z\left(\frac{P}{(a)} + \frac{Q}{(b)} - \frac{R}{(a)(b)}\right) = 1000\left(1 + \frac{(1-\lambda)P}{(a)} + \frac{(1-\lambda)Q}{(b)} + \frac{(\lambda-1)R}{(a)(b)}\right)$$

§. 16.

§. 15. Cum iam sit $\lambda = \frac{105}{100}$, erit $\lambda - 1 = \frac{5}{100}$ et $1000(\lambda - 1) = 50$, vnde nostra aequatio erit:

$$x + z\left(\frac{P}{(a)} + \frac{Q}{(b)} - \frac{R}{(a)(b)}\right) = 1000 - \frac{50P}{(a)} - \frac{50Q}{(b)} + \frac{50R}{(a)(b)}.$$

Quamobrem si totum pretium statim ab initio persolui debeat, ita vt sit $z = 0$, erit hoc pretium

$$x = 1000 - \frac{50P}{(a)} - \frac{50Q}{(b)} + \frac{50R}{(a)(b)}.$$

Sin autem velimus vt pretium per totum temporis intervallum vsque ad mortem vtriusque coniugis aequaliter distribuatur, poni debet $x = z$, atque contributio annua prodibit sequens:

$$z = \frac{1000 - \frac{50P}{(a)} - \frac{50Q}{(b)} + \frac{50R}{(a)(b)}}{1 + \frac{P}{(a)} + \frac{Q}{(b)} - \frac{R}{(a)(b)}}$$

sicque totum negotium huc redit, vt pro qualibet aetate vtriusque coniugis valores ternarum serierum litteris P, Q, R insignitarum inuestigentur, quos ergo in sequentibus euoluamus.

Euolutio valorum P et Q.

§. 16. Quoniam series Q simili modo ex aetate b definitur, quo series P ex aetate a erui debet, sufficiet alterutram tantum pro singulis aetatibus euoluisse. Cum igitur sit

$$P = \frac{(a+1)}{\lambda} + \frac{(a+2)}{\lambda^2} + \frac{(a+3)}{\lambda^3} \cdots\cdots + \frac{(95)}{\lambda^{95-a}}$$

si omnes termini huius seriei ad eandem denominationem λ^{95-a} reducantur, atque ordine retrogrado disponantur, fiet

$$P = \frac{1}{\lambda^{96-a}}\left((95) + (94)\lambda + (93)\lambda^2 + (92)\lambda^3 + \cdots\cdots + \lambda^{94-a}(a+1)\right).$$

§. 17. Euolutio autem huius ſeriei non parum foret taedioſa, ſi per ſingulos annos eam abſoluere vellemus. Quia autem valores characterum (a) et (b) non adeo ſunt certi, vt non aliquam aberrationem agnoſcere debeamus, ſufficiet quinos terminos ſe inſequentes inuicem coniungere eorumque ſummam quintuplo termini medii aequalem ſtatuere, ita vt pro quinis prioribus terminis ſcribi queat $5(93)\lambda^2$, quo facto valor noſtrae litterae P erit

$$P = \frac{5}{\lambda^{95-a}}((93)\lambda^2 + (88)\lambda^7 + (83)\lambda^{12} \ldots\ldots (a+3)\lambda^{92-a});$$

quare ſi hanc ſeriem littera p deſignemus, vt ſit

$$p = (93)\lambda^2 + (88)\lambda^7 + (83)\lambda^{12} + \ldots\ldots + (a+3)\lambda^{92-a}$$

inuento valore litterae p erit

$$P = \frac{5p}{\lambda^{95-a}}, \text{ hincque } \frac{P}{(a)} = \frac{5p}{(a)\lambda^{95-a}}.$$

Eodem modo, ſi ponatur

$$q = (93)\lambda^2 + (83)\lambda^7 + (83)\lambda^{12} \ldots\ldots + (b+3)\lambda^{92-b}$$

habebitur

$$Q = \frac{5q}{\lambda^{95-b}} \text{ et } \frac{Q}{(b)} = \frac{5q}{(b)\lambda^{95-b}}.$$

Euolutio tertii valoris R.

§. 18. Series quam littera R deſignauimus erat haec:

$$R = \frac{(a+1)(b+1)}{\lambda} + \frac{(a+2)(b+2)}{\lambda\lambda} + \ldots\ldots \frac{(a+n)(b+n)}{\lambda^n}$$

quam ſeriem eo vsque continuari oportet, donec termini ſequentes euaneſcant, quod fit, ſi alteruter numerorum $(a+n)$ vel $(b+n)$ ſuperet 95, vnde ſtatim ac maior horum duorum numerorum ad iſtum terminum exſurgit, ſeries hic terminata eſt cenſenda.

§. 19.

§. 19. Quia autem ambo numeri a et b in nostrum calculum aequaliter ingrediuntur, neque vllum discrimen inde nascitur, etiamsi hae duae litterae inter se permutentur, ita vt a denotet aetatem vxoris et b aetatem mariti: assumere poterimus aetatem a semper esse maiorem quam b; si enim vxor natu maior fuerit quam maritus, tum a designabit aetatem vxoris, at b mariti. Quare cum aetatem b tanquam minorem spectemus, discrimen littera d designemus, ita vt sit $b = a - d$, vbi quidem differentia d nulla erit, si ambo coniuges eandem habuerint aetatem.

§. 20. Hoc obseruato vltimus nostrae seriei terminus ibi erit, vbi fit $a + n = 95$ ideoque $n = 95 - a$, ita vt iam nostra series futura sit

$$R = \frac{(a+1)(b+1)}{\lambda} + \frac{(a+2)(b+2)}{\lambda^2} + \dots \frac{(95)(b+95-a)}{\lambda^{95-a}}$$

vbi ergo vltimus terminus est $\frac{(95)(95-d)}{\lambda^{95-a}}$. Reducamus nunc, vt ante, omnes has fractiones ad eandem denominationem λ^{95-a}, ac totam seriem ordine retrogrado disponamus, reperiemusque

$$R = \frac{1}{\lambda^{95-a}}((95)(95-d) + (94)(94-d)\lambda + (93)(93-d)\lambda^2$$
$$+ (92)(92-d)\lambda^3 + \dots \lambda^{94-a}(a+1)(a-d+1)),$$

cuius ergo seriei summam pro singulis valoribus amborum numerorum a et d computari oportet.

§. 21. Quo autem iste calculus facilior reddatur, iterum quinos terminos, vt ante fecimus, in vnum contrahamus; dum scilicet eorum summam quintuplo medii inter eos aequalem aestimabimus, quo facto habebimus

R =

$$R = \frac{5}{\lambda^{95-a}}((93)(93-d)\lambda^2 + (88)(88-d)\lambda^7 \ldots (a+3)(a+3-d)\lambda^{92-a}.$$

Quod si ergo ponamus

$r = (93)(93-d)\lambda^2 + (88)(88-d)\lambda^7 \ldots + (a+3)(a+3-d)\lambda^{92-a}$, inuento valore huius seriei r erit ipse valor, quem quaerimus $R = \frac{5r}{\lambda^{95-a}}$; et quoniam pro nostro calculo indigemus valore, $\frac{R}{(a)(b)}$ erit

$$\frac{R}{(a)(b)} = \frac{5r}{(a)(b)\lambda^{95-a}},$$

hisque valoribus pro singulis casibus inuentis aequatio nostra generalis erit

$$x + z\left(\frac{P}{(a)} + \frac{Q}{(b)} - \frac{R}{(a)(b)}\right) = 1000 - \frac{50P}{(a)} - \frac{50Q}{(b)} + \frac{50R}{(a)(b)}).$$

§. 22. Quoniam hic duo numeri occurrunt a et d, iste calculus multo maiorem laborem postulat quam praecedens pro seriebus P et Q, quem vt subleuemus, ambos numeros a et d per quinarium vel crescere vel decrescere assumemus, hanc ob rem plures casus euolui oportebit pro variis valoribus differentiae d, quam successiue statuemus 0, 5, 10, 15, 20, etc. Vnde hos casus sequenti modo ordine referamus.

I. Casus.

quo $d = 0$ ideoque $b = a$.

Hic ergo erit

$$r = (93)^2\lambda^2 + (88)^2\lambda^7 + (83)^2\lambda^{12} \ldots (a+3)^2\lambda^{92-a}$$

vnde fit

$$\frac{R}{(a)^2} = \frac{5r}{\lambda^{95-a}(a)^2}.$$

II.

II. Casus

quo $d = 5$ ideoque $b = a - 5$.

Tum ergo erit

$$r = (93)(88)\lambda^2 + (88)(83)\lambda^7 + (83)(78)\lambda^{12} \ldots (a+3)(a-2)\lambda^{92-a},$$

vnde fit

$$R = \frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-5)} = \frac{5r}{\lambda^{95-a}(a)(a-5)}.$$

III. Casus

quo $d = 10$ ideoque $b = a - 10$.

Hic ergo erit

$$r = (93)(83)\lambda^2 + (88)(78)\lambda^7 + (83)(73)\lambda^{12} \ldots (a+3)(a-7)\lambda^{92-a};$$

vnde fit

$$R = \frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-10)} = \frac{5r}{\lambda^{95-a}(a)(a-10)}.$$

IV. Casus

quo $d = 15$ ideoque $b = a - 15$.

Hoc casu erit

$$r = (93)(78)\lambda^2 + (88)(73)\lambda^7 + \ldots (a+3)(a-12)\lambda^{92-a},$$

vnde fit

$$R = \frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-15)} = \frac{5r}{\lambda^{95-a}(a)(a-15)}.$$

V. Casus

quo $d = 20$ ideoque $b = a - 20$.

Tum ergo erit

$$r = (93)(73)\lambda^2 + (88)(68)\lambda^7 + \ldots (a+3)(a-17)\lambda^{92-a};$$

vnde fit

$$R = \frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-20)} = \frac{5r}{\lambda^{95-a}(a)(a-20)}.$$

VI. Casus

quo $d = 25$ ideoque $b = a - 25$.

Hoc casu erit

$$r = (93)(68)\lambda^2 + (88)(63)\lambda^7 + (83)(58)\lambda^{12} \ldots (a+3)(a-22)\lambda^{92-a},$$

vnde fit

$$R = \frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-25)} = \frac{5r}{\lambda^{95-a}(a)(a-25)}.$$

VII. Casus

quo $d = 30$ ideoque $b = a - 30$.

Tum ergo erit

$$r = (93)(63)\lambda^2 + (88)(58)\lambda^7 + (83)(53)\lambda^{12} \ldots (a+3)(a-27)\lambda^{92-a}$$

vnde fit

$$R = \frac{5r}{\lambda^{95-a}} \text{ et } \frac{R}{(a)(a-30)} = \frac{5r}{\lambda^{95-a}(a)(a-30)}.$$

SOLVTIO QVARVNDAM QVAESTIONVM DIFFICILIORVM IN CALCVLO PROBABILIVM.

§. 1.

His quaestionibus occasionem dedit ludus passim publice institutus, quo ex nonaginta schedulis, numeris 1, 2, 3, 4 . . . . 90 signatis, statis temporibus quinae schedulae forte extrahi solent. Hinc ergo huiusmodi quaestiones oriuntur: quanta scilicet sit probabilitas vt, postquam datus extractionum numerus fuerit peractus, vel omnes nonaginta numeri exierint, vel saltem 89, vel 88, vel pauciores. Has igitur quaestiones, vtpote difficillimas, hic ex principiis calculi Probabilium iam pridem vsu receptis, resoluere constitui. Neque me deterrent obiectiones Illustris *D'Alembert*, qui hunc calculum suspectum reddere est conatus. Postquam enim summus Geometra studiis mathematicis valedixit, iis etiam bellum indixisse videtur, dum pleraque fundamenta solidissime stabilita euertere est aggressus. Quamvis enim hae obiectiones apud ignaros maximi ponderis esse debeant, haud tamen metuendum est, inde ipsi scientiae vllum detrimentum allatum iri.

§. 2. Qui in huiusmodi inuestigationibus elaborarunt, facile perspicient, resolutionem harum quaestionum calculos

culos maxime intricatos postulare, quos autem mihi beneficio certorum characterum, quibus iam aliquoties optimo successu sum vsus, superare licuit. Huiusmodi scilicet character: $[\frac{p}{q}]$, quo fractio vncinulis inclusa repraesentatur, mihi denotat istud productum :

$$\frac{p}{1} \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \frac{p-3}{4} \; : \; : \; : \; : \; : \; \frac{p-q+1}{q}$$

cuius ergo valor quouis casu facile exhiberi potest. Circa hunc characterem autem sequentia notasse iuuabit :

1°). Semper est $[\frac{p}{q}] = [\frac{p}{p-q}]$. 2°). Si $q = 0$ semper est $[\frac{p}{0}] = 1$. 3°). Si q sit vel numerus negatiuus, vel maior quam p, valor ipsius $[\frac{p}{q}]$ semper est $= 0$. 4°). Deinde si p numerus negatiuus, tum istam formulam : $-[\frac{p}{q}]$ reducere licet ad hanc : $\pm [\frac{p+q-1}{q}]$, vbi signum $+$ valet si q numerus par, $-$ vero si impar; vnde patet istam formam etiam in hanc mutari posse : $\pm [\frac{p+q-1}{p-1}]$.

§. 3. His praemissis quaestiones ex ludo memorato natas generalissime sum tractaturus. Numerum scilicet schedularum denotabo littera m, quas singulas litteris diuersis a, b, c, d, etc. signatas assumo, ne vsus numerorum absolutorum confusionem pariat. Deinde quouis tractu ex his schedulis i schedulas extrahi supponam, vnde numerus omnium variationum, quae in his tractibus contingere possunt, erit $= [\frac{m}{i}]$. Praeterea si numerus tractuum successiue institutorum fuerit $= n$, ex principiis combinationum patet, numerum omnium variationum, quae contingere queant, esse $[\frac{m}{i}]^n$. Hoc ergo modo sequentia Problemata percurram.

Pro-

Problema I.

Si numerus schedularum litteris a, b, c, d, etc. signatarum sit m, indeque quolibet tractu extrahantur i schedulae, atque iam numerus tractuum peractorum fuerit $= n$, quaeritur, quanta sit probabilitas vt omnes m litterae a, b, c, d, etc. exierint.

Solutio.

§. 4. Hic primo obseruandum est, quoniam in n tractibus numerus schedularum extractarum est in, omnes litteras exire non posse, nisi fuerit $in > m$, ideoque $n > \frac{m}{i}$, vel saltem non minus. Denotet iam Δ numerum omnium variationum, quae in his n tractibus euenire possunt, eritque vt iam indicauimus $\Delta = [\frac{m}{i}]^n$, qui cum sit numerus omnium casuum possibilium, pro nostra quaestione hinc omnes casus excludi debent, qui pauciores quam m litteras continent. Primo ergo, si numerus litterarum tantum esset $= m - 1$, quod m modis fieri potest, numerus casuum qui tantum $m - 1$ litteras, vel pauciores continent, erit $m [\frac{m-1}{i}]^n$, quem numerum ponamus $= A$. Simili modo, si binae litterae excludantur, quod $[\frac{m}{2}]$ modis fieri potest, numerus casuum tantum $m-2$, vel pauciores litteras continentium, erit $[\frac{m}{2}]. [\frac{m-2}{i}]^n$, quem numerum littera B indicemus. Porro sit C numerus omnium casuum, qui tantum $m-3$ litteras vel pauciores continent, eritque $C = [\frac{m}{3}]. [\frac{m-3}{i}]^n$. Eodemque modo fit

$$D = [\tfrac{m}{4}][\tfrac{m-4}{i}]^n; \; F = [\tfrac{m}{5}][\tfrac{m-5}{i}]^n; \text{ etc.}$$

Atque his elementis constitutis inueni numerum omnium casuum, qui omnes m litteras contineant, esse

$$\Delta - A + B - C + D - \text{etc.}$$

quem numerum indicemus littera Σ.

§. 5. Euidens est hunc numerum Σ per solam Theoriam combinationum determinari posse, ideoque nulli prorsus dubio esse obnoxium, ita vt tanquam veritas geometrica spectari possit. Hinc autem secundum Principia Probabilium numerus casuum fauorabilium per numerum omnium casuum possibilium diuisus praebebit probabilitatem quaesitam, quae ergo si ponatur $= \Pi$, erit $\Pi = \frac{\Sigma}{\Delta}$. Quare cum sit

$$\Sigma = \Delta - A + B - C + D - \text{etc.},$$

pro litteris Δ, A, B, C, etc. valores assignatos substituendo erit

$$\Sigma = [\tfrac{m}{i}]^n - [\tfrac{m}{1}][\tfrac{m-1}{i}]^n + [\tfrac{m}{2}][\tfrac{m-2}{i}]^n - [\tfrac{m}{3}][\tfrac{m-3}{i}]^n + \text{etc.}$$

Haec forma per $[\frac{m}{i}]^n$ diuisa dabit probabilitatem, quod post n tractus omnes m litterae exierint, vnde necessario ista expressio Σ semper nihilo aequalis esse debet, quoties fuerit $n < \frac{m}{i}$, quod etiam calculum pro casibus simplicioribus instituenti reuera euenire patebit. Veluti si fuerit $m = 7$, $n = 3$, $i = 2$ erit

$$[\tfrac{m}{i}]^n = 21^3 = 9261;$$
$$[\tfrac{m}{1}][\tfrac{m-1}{i}] = 7 \cdot 15^3 = 23625;$$
$$[\tfrac{m}{2}][\tfrac{m-2}{i}]^n = 21.\ 10^3 = 21000;$$
$$[\tfrac{m}{3}][\tfrac{m-3}{i}]^n = 35,\ 6^3 = 7560$$
$$[\tfrac{m}{4}][\tfrac{m-4}{i}]^n = 35 \cdot 3^3 = 945;$$
$$[\tfrac{m}{5}][\tfrac{m-5}{i}]^n = 21.\ 1^3 = 21;$$
$$[\tfrac{m}{6}][\tfrac{m-6}{i}]^n = 0;$$

vnde prodit $\Sigma = 0$.

§. 6. Dummodo ergo n non sit minus quam $\frac{m}{i}$ semper erit $\Sigma = 0$; at si fuerit $n = \frac{m}{i}$, siue $m = i\,n$, hic casus maxime est memorabilis; tum enim formula nostra pro Σ inuenta reduci potest ad productum ex meris factoribus constans. Erit enim

quod

$$\Sigma = \left[\frac{m}{i}\right] \cdot \left[\frac{m-i}{i}\right] \cdot \left[\frac{m-2i}{i}\right] \cdot \left[\frac{m-3i}{i}\right] \cdots \left[\frac{i}{i}\right];$$

quod vt exemplo illuſtremus, ſumamus vt ante $n = 3$ et $i = 2$, ſit vero $m = 6$, atque forma prior pro Σ data praebet $\Sigma = 90$, altera vero dat $\Sigma = 15 . 6 . 1 = 90$.

§. 7. Quanquam autem hae formulae, ſi pro n maiores numeri accipiantur, valde fiunt prolixae: tamen per logarithmos haud difficile erit quouis caſu valorem probabilitatis Π aſſignare. Cum enim ſit

$$\frac{A}{\Delta} = m\left[\frac{m-i}{m}\right]^n; \quad \frac{B}{A} = \frac{m-1}{2}\left[\frac{m-1-i}{m-1}\right]^n; \quad \frac{C}{B} = \frac{m-2}{3}\left[\frac{m-2-i}{m-2}\right]^n;$$

hinc ſumtis logarithmis erit

$$l\frac{A}{\Delta} = l\,m - n\,l\left[\frac{m}{m-i}\right];$$

$$l\frac{B}{A} = l\frac{m-1}{2} - n\,l\left[\frac{m-1}{m-1-i}\right];$$

$$l\frac{C}{B} = l\frac{m-2}{3} - n\,l\left[\frac{m-2}{m-2-i}\right];$$

etc.

ex quibus colligitur

$$l\frac{A}{\Delta} = l\,m - n\,l\left[\frac{m}{m-i}\right]$$

$$l\frac{B}{\Delta} = l\frac{A}{\Delta} + l\frac{m-1}{2} - n\,l\left[\frac{m-1}{m-1-i}\right]$$

$$l\frac{C}{\Delta} = l\frac{B}{\Delta} + l\frac{m-2}{3} - n\,l\left[\frac{m-2}{m-2-i}\right]$$

etc.

Vnde ergo facile inueniuntur valores $\frac{A}{\Delta}$, $\frac{B}{\Delta}$, $\frac{C}{\Delta}$, etc. quibus inuentis probabilitas quaeſita erit

$$\Pi = 1 - \frac{A}{\Delta} + \frac{B}{\Delta} - \frac{C}{\Delta} + \frac{D}{\Delta} - \text{etc.}$$

§. 8. Applicemus haec ad caſum ludi initio memorati, quo eſt $m = 90$ et $i = 5$, eritque vt ſequitur:

$$l\frac{A}{\Delta} = l\,90 - n\,l\frac{90}{85} = 1,9542425 - n \cdot 0,0248236$$

$$l\frac{B}{\Delta} = l\frac{A}{\Delta} + l\frac{89}{2} - n\,l\frac{89}{84} = l\frac{A}{\Delta} + 1,6483600 - n \cdot 0,0250107$$

$l\frac{B}{\Delta}$

$l\frac{C}{\Delta} = l\frac{B}{\Delta} + l\frac{88}{3} - n\,l\frac{88}{83} = l\frac{B}{\Delta} + 1,4673614 - n.\ 0,0254046$

$l\frac{D}{\Delta} = l\frac{C}{\Delta} + l\frac{87}{4} - n\,l\frac{87}{82} = l\frac{C}{\Delta} + 1,3374593 - n.\ 0,0257054$

$l\frac{E}{\Delta} = l\frac{D}{\Delta} + l\frac{86}{5} - n\,l\frac{86}{81} = l\frac{D}{\Delta} + 1,2355283 - n.\ 0.0260133$

$l\frac{F}{\Delta} = l\frac{E}{\Delta} + l\frac{85}{6} - n\,l\frac{85}{80} = l\frac{E}{\Delta} + 1,1512676 - n.\ 0,0263289$

$l\frac{G}{\Delta} = l\frac{F}{\Delta} + l\frac{84}{7} - n\,l\frac{84}{79} = l\frac{F}{\Delta} + 1,0791813 - n.\ 0,0266522$

etc. etc.

§. 9. Perspicuum hic est, quo maior accipiatur numerus tractuum n, eo promptius istam progressionem convergere, ita vt, si n denotet numerum vehementer magnum, semper proxime prodituram sit $\Pi = 1$: tum scilicet maxime erit probabile omnes prorsus m numeros exiisse. Contra autem, si numerus n parum superet minimum valorem $\frac{m}{t} = 18$, euolutio horum terminorum maxime fiet operosa, cum pluribus terminis sit opus, ante quam ad euanescentes perueniatur. Sumamus $n = 100$, vt huic quaestioni respondeamus: quanta sit probabilitas, vt post centum tractus omnes nonaginta numeri exierint? Hic ergo erit

$l\frac{A}{\Delta} = 9,47188$, ergo $\frac{A}{\Delta} = 0,2964$

$l\frac{B}{\Delta} = 8,61917$ — $\frac{B}{\Delta} = 0,0416$

$l\frac{C}{\Delta} = 7,54607$ — $\frac{C}{\Delta} = 0,0035$

$l\frac{D}{\Delta} = 6,31299$ — $\frac{D}{\Delta} = 0,0002$

$l\frac{E}{\Delta} = 4,94719$ — $\frac{E}{\Delta} = 0,0000$

ergo $\Pi = 0,7419$.

§. 10. Sit $n = 200$ et pro hoc casu erit

$l\frac{A}{\Delta} = 6,98952$, ergo $\frac{A}{\Delta} = 0,00097$

$l\frac{B}{\Delta} = 3,63574$, ergo $\frac{B}{\Delta} = 0,00000$

vnde

vnde colligitur probabilitas, quod post 200 extractiones omnes numeri exierint, $\Pi = 0,99903$, quae probabilitas certitudini omnes exiisse valde est propinqua.

Problema II.

Positis quae in Problemate praecedente sunt constituta quaeritur, quanta futura sit probabilitas, vt saltem $m-1$ litterae post n tractus exierint.

Solutio.

§. 11. Hic ergo numerus tractuum omnes m litteras continentium non excluditur, vnde patet tractuum numerum nostro praesenti casu fore maiorem. Calculo autem subducto, si numerus horum casuum ponatur Σ', inueni fore

$$\Sigma' = \Delta - B + 2\,C - 3\,D + 4\,E - 5\,F + \text{etc.}$$

vnde probabilitas, quod post n tractus saltem $m-1$ litterae exierint, erit $\Pi' = \frac{\Sigma}{\Delta}$, ideoque

$$\Pi' = 1 - \frac{B}{\Delta} + 2\frac{C}{\Delta} - 3\frac{D}{\Delta} + 4\frac{E}{\Delta} - \text{etc.}$$

§. 12. Hoc ergo casu erit

$$\Sigma' = \left[\frac{m}{1}\right]^n - \left[\frac{m}{2}\right]\left[\frac{m-2}{1}\right]^n + 2\left[\frac{m}{3}\right]\left[\frac{m-3}{1}\right]^n - \left[\frac{m}{4}\right]\left[\frac{m-4}{1}\right]^n + \left[\frac{m}{5}\right]\left[\frac{m-4}{1}\right]^n - \text{etc.}$$

Vnde si applicatio fiat ad ludum memoratum, cum litterae Δ, A, B, C, D, etc. eosdem retineant valores, calculus per logarithmos institutus ex inuentis valoribus $\frac{B}{\Delta}$, $\frac{C}{\Delta}$, $\frac{D}{\Delta}$, etc. facile perficietur. Ita si post 100 tractus requiratur probabilitas quod saltem 89 numeri exierint, ob

$$\frac{B}{\Delta} = 0,0416;\quad \frac{C}{\Delta} = 0,0035;\quad \frac{D}{\Delta} = 0,0002;$$

erit ista probabilitas $\Pi' = 0,9648$. Vnde sequitur probabilitatem, quod tantum pauciores numeri exierint, fore $0,0352$.

Problema III.

Iisdem positis, vt hactenus, quaeritur: quanta sit probabilitas, vt saltem $m-2$ litterae post n tractus fuerint extractae.

Solutio.

§. 13. Numerus omnium casuum, qui saltem $m-2$ litteras contineant, per litteras ante stabilitas Δ, A, B, C, D, etc. ita definitur, vt sit

$$\Sigma'' = \Delta - C + 3D - 6E + 10F - \text{etc.}$$

quae expressio, restitutis valoribus, hoc modo se habet:

$$\Sigma'' = \left[\tfrac{m}{1}\right]^n - \left[\tfrac{2}{2}\right]\left[\tfrac{m}{2}\right]\left[\tfrac{m-2}{1}\right]^n + \left[\tfrac{3}{2}\right]\left[\tfrac{m}{3}\right]\left[\tfrac{m-3}{1}\right]^n - \left[\tfrac{4}{2}\right]\left[\tfrac{m}{4}\right]\left[\tfrac{m-4}{1}\right]^n$$

atque hinc probabilitas erit

$$\Pi = 1 - \frac{C}{\Delta} + 3\frac{D}{\Delta} - 6\frac{E}{\Delta} + 10\frac{F}{\Delta} - \text{etc.}$$

Pro ludo igitur ante memorato, si quaeratur probabilitas, vt post 100 tractus saltem 88 numeri exierint, ea reperietur $\Pi'' = 0,9971$, vnde probabilitas, quod contrarium euenit, erit $= 0,0029$.

Problema generale.

Iisdem positis vt ante, quaeritur quanta sit probabilitas, vt post n tractus saltem $m-\lambda$ litterae exierint.

Solutio.

Numerus casuum ad minimum tot litteras continentium per nostros characteres ita commode exprimitur, vt sit

$\left[\tfrac{m}{1}\right]^n$

$$\left[\frac{m}{i}\right]^n - \left[\frac{\lambda}{\lambda}\right]\left[\frac{m}{1+\lambda}\right]\left[\frac{m-1-\lambda}{i}\right]^n + \left[\frac{\lambda+1}{\lambda}\right]\left[\frac{m}{\lambda+2}\right]\left[\frac{m-\lambda-2}{i}\right]^n$$
$$-\left[\frac{\lambda+2}{\lambda}\right]\left[\frac{m}{\lambda+3}\right]\left[\frac{m-\lambda-3}{i}\right]^n + \text{etc.}$$

quae formula per terminum primum $\left[\frac{m}{i}\right]^n$ diuisa praebebit probabilitatem quaesitam.

§. 15. In his probabilitatibus aestimandis vtique assumitur omnes litteras ad extrahendum aeque esse procliues, quod autem Ill. *D'Alembert* negat assumi posse. Arbitratur enim, simul ad omnes tractus iam ante peractos respici oportere; si enim quaepiam litterae nimis crebro fuerint extractae, tum eas in sequentibus tractibus rarius exituras; contrarium vero euenire si quaepiam litterae nimis raro exierint. Haec ratio, si valeret, etiam valitura esset si sequentes tractus demum post annum, vel adeo integrum saeculum, quin etiam si in alio quocunque loco instituerentur; atque ob eandem rationem etiam ratio haberi deberet omnium tractuum, qui iam olim in quibuscunque terrae locis fuerint peracti, quo certe vix quicquam absurdius excogitari potest.

Demonstratio solutionum praecedentium.

§. 16. Cum numerus omnium litterarum a, b, c, d, etc. quibus singulas schedulas signatas assumimus, sit $= m$, hunc litterarum complexum vocabo systema principale, vnde alia systemata deriuata, quae pauciores litteras contineant, formari conueniet, quae ita in ordines dispesco, vt ordo primus complectatur omnia systemata quae tantum $m-1$ litteras contineant, quorum ergo numerus erit $= m$. Ad ordinem vero secundum referam omnia systemata in quibus litterarum numerus est $m-2$, quorum numerus erit $\frac{m}{1}.\frac{m-1}{2} = \left[\frac{m}{2}\right]$. Ordo autem tertius habebit omnia systemata, vbi numerus litte-

litterarum est $m-3$, quorum numerus est $\frac{m}{1}\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}=[\frac{m}{3}]$. Eodem modo numerus systematum quarti ordinis tantum $m-4$ litteras continentium erit $[\frac{m}{4}]$; quinti autem ordinis, vbi tantum $m-5$ litterae insunt, numerus systematum erit $[\frac{m}{5}]$, et ita porro.

§. 17. Quae quo fiant clariora, systema principale his sex litteris *a b c d e f* constans contemplemur, ex quo ergo sequentia deriuata cuiusque ordinis resultabunt, quae hac tabula exhibemus:

I.	II.	III.	IV.	V.
a b c d e	*a b c d*	*a b c*	*a b*	*a*
a b c d f	*a b c e*	*a b d*	*a c*	*b*
a b c e f	*a b c f*	*a b e*	*a d*	*c*
a b d e f	*a b d e*	*a b f*	*a e*	*d*
a c d e f	*a b d f*	*a c d*	*a f*	*e*
b c d e f	*a b e f*	*a c e*	*b c*	*f*
	a c d e	*a c f*	*b d*	
	a c d f	*a d e*	*b e*	
	a c e f	*a d f*	*b f*	
	a d e f	*a e f*	*c d*	
	b c d e	*b c d*	*c e*	
	b c d f	*b c e*	*c f*	
	b c e f	*b c f*	*d e*	
	b d e f	*b d e*	*d f*	
	c d e f	*b d f*	*e f*	
		b e f		
		c d e		
		c d f		
		c e f		
		d e f		

vbi

vbi ergo numerus systematum ordinis primi est $6 = [\frac{6}{1}]$, ordinis secundi $15 = [\frac{6}{2}]$, ordinis tertii $20 = [\frac{6}{3}]$, quarti $= 15 = [\frac{6}{4}]$, quinti $= 6 = [\frac{6}{5}]$.

§. 18. Nunc euidens est singula systemata cuiusque ordinis inferioris in omnibus superioribus contineri, quod quoties eueniat plurimum interest obseruasse. Ita pro casu $m = 6$ systema primi ordinis $a\,b\,c\,d\,e$ in ordine hoc semel occurrit. At systema secundi ordinis $a\,b\,c\,d$ in primo ordine bis, in secnndo semel occurrit. Deinde systema tertii ordinis $a\,b\,c$ in primo ordine ter, in secundo ter, at in tertio semel reperitur. Systema quarti ordinis $a\,b$ in primo ordine quater, in secundo sexies, in tertio quater, in quarto semel inest. Denique systema quinti ordinis in primo ordine quinquies occurrit, in secundo decies, in tertio decies, in quarto quinquies, in quinto semel. Ex quo manifestum est hos numeros conuenire cum coëfficientibus binomii ad potestates eleuato, siquidem omnia systemata in ipso principali semel continentur.

§. 19. Hinc ergo in genere pro quouis systemate cuiuspiam ordinis inferioris facile assignari potest, quot modis in quolibet ordine superiore occurrat, id quod sequens tabula manifesto declarabit, vbi systema principale littera O, systemata autem primi, secundi, tertii, quartii, etc. ordinis notis romanis I, II, III, IV V, VI denotabo.

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	O	I	II	III	IV	V	VI
m	1						
$m-1$	1	1					
$m-2$	1	2	1				
$m-3$	1	3	3	1			
$m-4$	1	4	6	4	1		
$m-5$	1	5	10	10	5	1	
$m-6$	1	6	15	20	15	6	1
— —	—	—	—	—	—	—	—
— —	—	—	—	—	—	—	—
— —	—	—	—	—	—	—	—
$m-\lambda$	$[\frac{\lambda}{0}]$	$[\frac{\lambda}{1}]$	$[\frac{\lambda}{2}]$	$[\frac{\lambda}{3}]$	$[\frac{\lambda}{4}]$	$[\frac{\lambda}{5}]$	$[\frac{\lambda}{6}]$

§. 20. Conſideremus nunc numerum ſchedularum, quae quouis tractu tam ex ſyſtemate principali actu extrahuntur, quam ex ſyſtematibus deriuatis extrahi concipi poſſunt, quae quidem ex principali facillime deduci poterunt. Quodſi quouis tractu vnica littera extrahatur, pro ſyſtemate principali numerus tractuum diuerſorum erit $=[\frac{m}{1}]$; ſin autem binae litterae ſimul extrahantur, numerus omnium tractuum diuerſorum erit $\frac{m}{1}\cdot\frac{m-1}{2}=[\frac{m}{2}]$. Si ternae litterae quouis tractu extrahantur, numerus tractuum diuerſorum erit $[\frac{m}{3}]$, atque in genere, ſi i litterae quouis tractu extrahantur, numerus omnium tractuum diuerſorum erit $[\frac{m}{i}]$. Sin autem tales extractiones etiam ex ſyſtematibus deriuatis fieri concipiantur, pro quolibet ſyſtemate ordinis primi numerus tractuum diuerſorum erit $[\frac{m-1}{i}]$, ordinis ſecundi $=[\frac{m-2}{i}]$, ordinis tertii $[\frac{m-3}{i}]$, et ita porro.

§. 21. Quodſi iam hae extractiones bis repetantur, quoniam pro ſyſtemate principali quemlibet tractum non

ſolum

ſolum reliquae omnes ſequi poſſunt, ſed etiam ipſae, numerus diuerſorum caſuum erit $[\frac{m}{i}]^2$. Si tres extractiones ſucceſſiue inſtituantur, omnium caſuum numerus erit $[\frac{m}{i}]^3$; atque in genere, ſi n extractiones ſibi ſuccedant, numerus omnium caſuum poſſibilium erit $[\frac{m}{i}]^n$, quem numerum littera Δ ſupra deſignauimus, ita vt ſit $\Delta = [\frac{m}{i}]^n$.

§. 22. Simili modo numerus omnium caſuum, qui in quolibet ſyſtemate primi ordinis locum habere poſſunt, eſt $[\frac{m-1}{i}]^n$; quare cum horum ſyſtematum numerus ſit $[\frac{m}{1}]$, numerus omnium caſuum, quem primus ordo praebet, erit $[\frac{m}{1}][\frac{m-1}{i}]^n$, quemque littera A deſignemus, ita vt ſit $A = [\frac{m}{1}][\frac{m-1}{i}]^n$. Eodem modo facile intelligitur, numerum omnium caſuum, qui ex ſingulis ſyſtematibus oriri poſſunt eſſe

pro ordine ſecundo $B = [\frac{m}{2}][\frac{m-2}{i}]^n$,

pro ordine tertio $C = [\frac{m}{3}][\frac{m-3}{i}]^n$,

pro ordine quarto $D = [\frac{m}{4}][\frac{m-3}{i}]^n$,

et ita porro. His iam praemiſſis ſolutiones ſingulorum Problematum praecedentium facile expedire licebit.

Pro Problemate primo.

§. 23. Cum in hoc Problemate ex omnibus caſibus poſſibilibus, quorum numerus eſt Δ, ii enumerari debeant, qui omnes m litteras inuoluunt, inde excludamus primo omnes caſus, qui tantum $m-1$ litteras, vel pauciores continent, quod fiet ſi omnes caſus poſſibiles primi ordinis, quorum numerus eſt A, auferamus. Hoc enim modo caſus, qui $m-1$ litteras continent, e medio tollentur. At vero caſus qui $m-2$ litteras continent, bis aufferentur hoc modo;

vnde

vnde in formula $\Delta - A$ ſemel deficient, ita vt eorum numerus $1 - 2 = - 1$. At pro caſibus $m - 3$ litteras continentibus numerus, quo in formula $\Delta - A$ occurrent, erit $1 - 3 = - 2$. Simili modo pro $m - 4$ habebimus $1 - 4 = - 3$, et ita porro, qui ergo caſus deficientes iterum reſtitui debebunt.

§. 24. Caſus autem formae $m - 2$ ſemel deficientes reſtituentur, ſi ad formulam $\Delta - A$ addatur B. Hoc autem modo terminos formae $m - 3$ ter adduntur, cum tamen bis tantum deficiſſent, ergo nunc ſemel abundabunt, ſiue index erit $+1$. At forma $m - 4$ ſexies adiicitur, cum tantum ter defuiſſet, ideoque index erit $+ 3$. Simili modo pro terminis formae $m - 5$ index erit $10 - 4 = + 6$, et ita porro.

§. 25. Vt igitur hos caſus iam abundantes iterum tollamus, ſubtrahamus omnes caſus ordinis tertii $= C$. Hoc enim modo termini formae $m - 3$ penitus tollentur, reliqui autem nimis crebro aufferentur, ſcilicet pro ordine $m - 4$ index erit $- 1$, pro ordine $m - 5$, index erit $- 4$, etc.

§. 26. Quia forma $m - 4$ ſemel deficit, reſtitutio fiet addendo litteram D. Inferiores autem nunc redundabunt ſecundum indices 1, 5, 15, etc. vnde E ſubtrahendo hi tollentur; quod minis ſubtractum eſt additione litterae F reſtituetur, et ita porro.

§. 27. Hinc iam ſatis manifeſtum eſt, ex forma Δ ſublatos eſſe omnes caſus pauciores quam m litteras continentes, quorum ergo reſtantium numerus erit

$$\Delta - A + B - C + D - E + F - \text{etc.}$$

quem

quem indicauimus per Σ, sicque Solutio primi Problematis firmiter est demonstrata.

Pro Problemate secundo.

§. 28. Manifestum est, secundum idem, quo hic usi sumus, ratiocinium procedendo, demonstrationem pro secundi Problematis solutione adornari posse. Nec opus erit omnia tam prolixe exponere. Cum enim ex numero casuum possibilium Δ ii sint enumerandi, qui tantum $m-1$ litteras continent, statim patet, hinc excludendos esse omnes casus $m-2$ litteras continentes, quod fiet si a numero Δ numerus B subtrahatur. At tabula supra § 19 data declarat, hoc modo terminos formae $m-3$ ter ablaturos esse, cum tamen semel tantum subtrahi debuissent, et ita de reliquis formis. Ad eos restituendos addatur numerus $2\,C$, quo numeri deficientes formae $m-3$ penitus tollentur: redundabunt autem numeri formae $m-4$, indice 3; ac magis superfluunt sequentes. Quo priores tollantur iterum subtrahi debet numerus $3\,D$, quo sublato termini formae $m-4$ exclusi erunt. Deficientes numeri formae $m-5$ et sequentium iterum additione numeri $4\,E$ erunt restituendi, et ita porro, quibus operationibus peractis numerus restantium erit

$$\Sigma' = \Delta - B + 2C - 3D + 4E - 5F + \text{ etc.}$$

sicque solutio secundi Problematis est demonstrata.

Pro Problemate tertio.

§. 29. Hic a numero Δ subtrahi debet numerus C, quo casus $m-3$ schedularum excludantur, et quia hoc pacto numerus $m-4$ quater subtrahitur, cum tantum semel

mel redundabat, iterum addi debet numerus 3 D, quo ille et ſequentes deficientes reſtituantur. Quod excedit ſubtractione numeri 6 E tolletur: deficientes vero additione numeri 10 F reſtituendi ſunt, et ita porro, vnde numerus caſuum $m-2$ litteras continentium erit

$$\Sigma'' = \Delta - C + 3D - 6E + 10F - \text{etc.}$$

quemadmodum in ſolutione tertii Problematis aſſeueraui. Hoc modo igitur haec etiam ſolutio firmiter eſt demonſtrata.

Fig. 1.

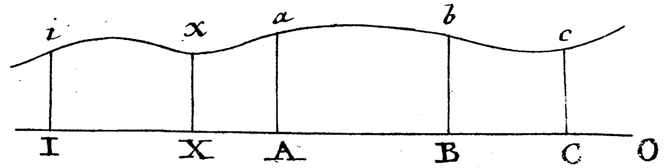


Fig. 2.

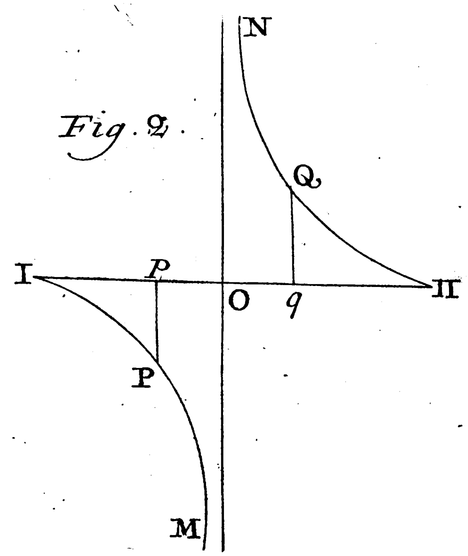


Fig. 3.

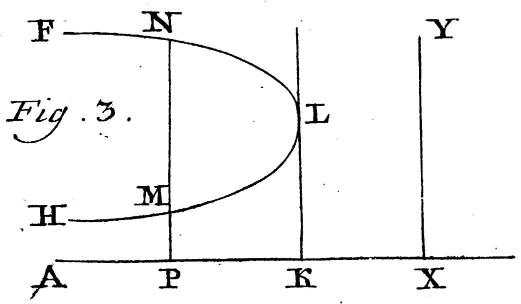


Fig. 4.

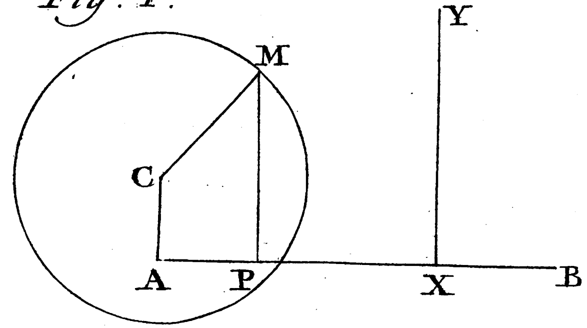


Fig. 6.

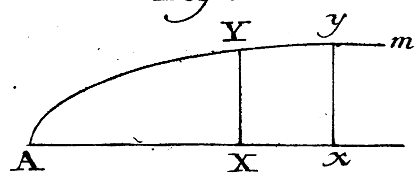
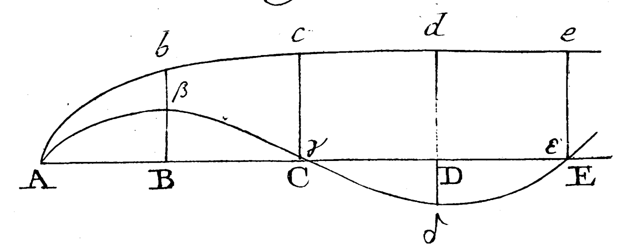


Fig. 6.



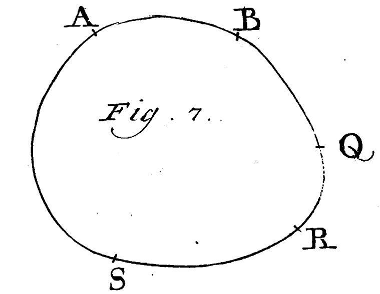


Fig. 7.

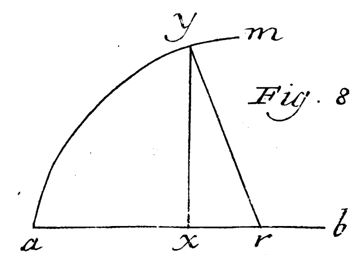


Fig. 8.

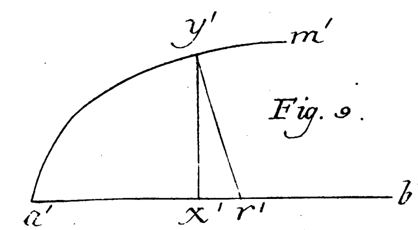


Fig. 9.

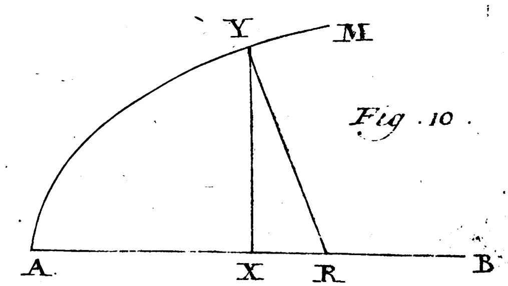


Fig. 10.

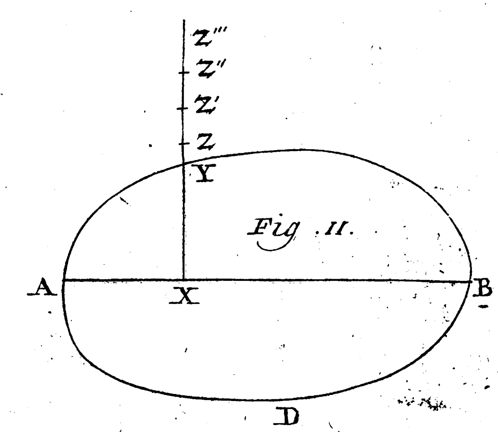


Fig. 11.